A Theory of Endogenous Asset Fire Sales, Bank Runs, and Contagion*

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Abstract

In a global-games framework, we endogenize asset fire sales, bank runs, and contagion by emphasizing a lack of information: investors can be uncertain whether banks selling assets to fend off runs are insolvent or simply illiquid. However, it is this uncertainty that leads to asset price collapses and runs in the first place. We show that a balanced-budget asset purchase program promotes financial stability by breaking down this vicious cycle. By contrast, increasing capital can exacerbate fire sales in the presence of adverse selection, because runs on well-capitalized banks signal high risks. We also derive implications regarding regulatory disclosure policies.

Keywords: Bank run, Global games, Asymmetric information, Capital, Asset purchase program

JEL Classification: G01, G11, G21

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1 Introduction

The recent banking crisis highlights the importance of liquidity risk. During the crisis, market liquidity evaporated, and asset prices dropped sharply. At the same time, as funding liquidity dried up, even well-capitalized banks found it difficult to roll over their short-term debt and had to resort to central banks.

Market liquidity and funding liquidity are closely related. It is widely acknowledged that market illiquidity contributes to funding illiquidity. As market liquidity diminishes, potential fire-sale losses from early liquidation make creditors panic. As a result, the creditors can stop rolling over their short-term debt and thus deprive a healthy financial institution of its funding. Such funding liquidity risk has been emphasized by, among others, Morris and Shin (2000), Rochet and Vives (2004), and Goldstein and Pauzner (2005). However, the literature has so far ignored the fact that funding illiquidity also feeds market illiquidity: bank runs can lead to fire sales, depress asset prices, and in extreme cases, freeze up markets.[1]

The current paper fills this gap by presenting a model where asset fire sales and bank runs mutually reinforce each other. The feedback is driven by a lack of information. In particular, a bank that fails because of illiquidity cannot be distinguished from one that fails because of fundamental insolvency.[2] We incorporate this informational friction into a global-games-based bank run model and endogenize fire-sale prices: asset prices are based on the micro-foundation of information asymmetry. As a defining feature that distinguishes the current model from the literature, we have market participants' beliefs, asset prices, and bank runs, all endogenous and jointly determined in a rational expectations equilibrium. We prove that an equilibrium exists and is unique. We also extend the theoretical framework to model financial contagion. We show that contagion can be driven by creditors' beliefs and emerge as a result of multiple equilibria.

The intuition behind our model is as follows. When asset buyers cannot distinguish assets sold by an illiquid bank from those sold by an insolvent one, their offered price will be distorted downwards. As a result, the illiquid bank cannot recoup a fair value for its assets on sale. This friction leads to a vicious circle. To begin with, low asset prices fuel bank runs: expecting

[1] For example, Acharya and Roubini (2009) documented how the early liquidation of two highly levered Bear Stearns-managed hedge funds stressed the price of asset-backed securities.

[2] This idea is widely recognized in practice as well as in the literature. In fact, it is considered the main challenge for central banks to perform as lenders of last resort. See, for example, Freixas, Rochet, and Parigi (2004).

other creditors' early withdrawals to cause fire-sale losses, a creditor has an incentive to join the run. However, it is the run and early liquidation, which pools illiquid banks with insolvent ones, that leads to the drop in asset price in the first place. In this sense, creditors' pessimistic expectations realize themselves. With one more ingredient—an aggregate state — we show that financial contagion can happen in a similar manner. As uninformed asset buyers form rational expectations, they revise their beliefs about the aggregate state downwards if they observe any bank runs. Their deteriorating expectations reduce the asset price that they are willing to pay, which, in turn, precipitates contagious runs to other banks.

Our model delivers three policy insights. First, while our paper confirms that well-capitalized banks have larger buffers against fire-sale losses, our analysis also reveals that once asset prices are endogenous, increasing capital can also have unintended consequences for liquidity via buyers' beliefs. In particular, buyers' posterior beliefs about a bank's asset value will deteriorate when a run happens to the bank, and the deterioration is particularly strong when the bank maintains a high capital ratio. Because well-capitalized banks can sustain large losses, if a run happens to such a bank, the bank's losses must be unusually high. Therefore, given that a bank faces a run, buyers' valuation of its assets decreases in its capital level. Asset buyers' low willingness to pay, however, makes the run more likely to happen in the first place. In the extreme, the model predicts that increasing capital does not reduce bank funding liquidity risk at all.

Second, our model highlights the effectiveness of asset purchase programs in promoting financial stability. We find that even when regulators have the same information as other market participants, it is still possible to design an asset purchase program that improves financial stability at no social cost. In such an asset purchase program, the regulator pre-commits to buy bank assets for a price at which she expects to break even. This precommitment breaks the vicious cycle fueled by beliefs. It can thus be seen that the typical asset buyers' lack of commitment is at the very root of financial fragility. In particular, typical asset buyers behave according to their rational beliefs, and aim to avoid losses in every perceived state. This generates a vicious cycle because buyers' pessimistic beliefs can lead to negative market outcomes (e.g., more bank runs) which in turn justify themselves. A regulator with commitment power, on the other hand, avoids such pessimistic belief updating. This allows her to promote financial stability while breaking even from an ex-ante perspective.

Finally, when the regulator has better information than typical market participants, the model highlights a trade-off for regulatory disclosure policies.[3] Our paper shows that regulatory disclosures are a double-edged sword. If the disclosed information reassures market participants, banks can be saved from illiquidity. However, if the disclosure adds to pessimistic market beliefs, the disclosure itself can lead to financial fragility. This is because once the severity of the problem is acknowledged, market participants will further revise downwards their expected performance of all banks, leading to greater fire-sale losses and triggering illiquidity, even for healthy banks.

Our theoretical framework is related to the literature on bank runs and financial contagion. Since Diamond and Dybvig (1983), the literature has been concerned with the financial fragility caused by bank runs.[4] Following their seminal contribution, there was a debate as to whether bank runs are due to pure panic or unfavorable information on banks' fundamentals.[5] The gap between the panic and fundamental view was bridged by the application of global games. Using the concept, papers such as Morris and Shin (2000), Rochet and Vives (2004) and Goldstein and Pauzner (2005) refined the multiple equilibria in Diamond and Dybvig (1983) and emphasized the role of fire-sale losses in causing bank runs. That is, to prevent runs, an extra buffer of cash flow is needed against fire-sale losses. A bank that fails to provide the extra buffer will become "solvent but illiquid"—being able to repay its debt in full if no run happens, but to be liquidated early if its creditors do not roll over their debt. But a limitation of the existing models is that they build on the simplifying assumption of exogenous fire-sale losses,[6] so that the models implicitly exclude the possibility for bank runs to affect asset prices. In contrast, the current paper explores the relationship: when it is difficult to distinguish illiquid banks from insolvent ones, the adverse selection causes low asset prices and fire-sale losses.[7]

A natural corollary of assuming an exogenous fire-sale loss is that funding liquidity risk will be always reduced by higher capital, because the market value of capital adds to the buffer

[3]For instance, whether or not regulators should disclose information concerning their assistance programs.

[4]It should be mentioned that some papers also consider the positive role of bank run as disciplinary device: Calomiris and Kahn (1991) and Diamond and Rajan (2001).

[5]Papers emphasizing banks' weak fundamentals in causing runs include Chari and Jagannathan (1988), Jacklin and Bhattacharya (1988) and Allen and Gale (1998). Friedman and Schwartz (1963) provide empirical support for the panic view. Contradicting evidence in favor of the fundamental view is present in Gorton (1988), Calomiris and Gorton (1991) and Calomiris and Mason (2003).

[6]For example, Rochet and Vives (2004) assume an exogenous fire-sale discount and Morris and Shin (2009) assume an exogenous haircut.

[7]While the current paper justifies the low asset price by informational frictions, low asset prices can also be explained by fixed short-term cash supply—the cash-in-the-market argument pioneered by Shleifer and Vishny (1992) and Allen and Gale (1994).

against fire-sale losses. With endogenous fire-sale prices, the current paper takes a broader view: while acknowledging the buffer effect of capital, we point out that greater capital can also contribute to illiquidity via buyers' pessimistic inference.

In terms of predicting an interaction between market liquidity and funding liquidity, our model is most closely related to Brunnermeier and Pedersen (2009), who emphasize a haircut constraint on a speculator who supplies liquidity to a financial market with limited participation. In their model, asset prices are volatile because there is an asynchronization between selling and buying. This paper differs from theirs in two aspects. First, the funding liquidity risk arises as a result of equilibrium bank runs caused by the wholesale creditors' coordination failures. Second, this paper emphasizes the asymmetric information on asset qualities, and how such adverse selection causes asset illiquidity.[8] .

In our paper, contagion is generated not only by the actual realization of a common risk factor but also by its perception: a bank failure casts shadow on the perceived common risk factor; and the resulting negative informational externalities affect all the other banks. This observation is mostly related to the literature of information contagion, as exemplified by Acharya and Thakor (2011) and Oh (2012).[9] Compared to the existing work, the current paper emphasizes the self-fulfilling nature of such contagion and the two-way feedback between runs and asset fire sales.

On the impact of increasing bank capital, the current paper relates to a few models that show increased capital requirements can increase bank risk. Martinez-Miera (2009) argues that equity increases banks' cost of funding, which leads to higher loan rates and spurs risk-taking by borrowers. As a result, banks' portfolio risk rises passively. Hakenes and Schnabel (2007) argue that a higher capital requirement erodes charter value and induces banks to actively take risks; when the higher capital requirement decreases credit supply, it also leads to borrower risk-taking via a hike in the loan rate. What all these papers have in common is that they focus on insolvency risk. To the best of our knowledge, the current study is the first to show that capital can contribute to the risk of bank runs.

[8]Using historical data, Fohlin, Gehrig, and Haas (2016) empirically document the feedback between market and funding illiquidity, and provide supporting evidence that adverse selection on asset qualities contributes to the vicious cycle.

[9]There are other approaches to model contagion. For instance, Freixas and Parigi (1998) and Freixas, Parigi, and Rochet (2000) model direct linkages of banks through payment systems. Allen and Gale (2000) emphasize the role of interbank market. Gorton (1988) studies how banks' common risk exposures directly contribute to systemic risk.

On asset purchase programs, our paper is mostly related to the empirical policy evaluation in Veronesi and Zingales (2010). The authors used Black-Scholes-Merton model to evaluate the ex-ante costs and benefits of "Paulson's Plan", and concluded that the intervention yielded "a net benefit between \$86 and \$109 bn". The policy evaluation in our theoretical model also takes an ex-ante perspective, and confirms the effectiveness of asset purchase programs in promoting financial stability.

The discussion on disclosure policy is most related to several recent papers on the instability consequences of public signals: Morrison and White (2013) concern that a public bailout can reveal regulatory deficiency and make market participants lose their confidence in all other banks under the same regulation. Dang, Gorton, and Holmström (2010) show that a public signal could increase adverse selection to debt-like securities that are otherwise information insensitive. And Wang (2013) empirically documents that after individual banks were identified in Trouble Asset Relief Program (TARP), bank run probabilities, as reflected in CDS spread and stock market abnormal returns, rose dramatically—an outcome the author attributes to the bad news nature of public bailout. Our paper abstracts from specific policy announcements and shows that as long as market participants believe the regulator is better informed, any regulatory action or announcement concerning banks' common risk exposure may generate financial contagion.

The paper proceeds as follows. Section 2 lays out the model. Section 3 presents the baseline bank-run model with endogenous fire-sale prices. With only one aggregate state, the baseline model delivers a closed-form solution and allows us to discuss the first policy issue such as whether higher capital can effectively reduce the risk of bank runs. In Section 4, we analyze contagion in a fully-fledged model with two banks and two aggregate states. We show that even if a regulator has no better information than typical market participants, asset purchase programs can still improve financial stability at no social cost. In Section 5, we discuss the trade-off associated with regulatory disclosures. And section 6 concludes.

2 Model setup

We consider a three-date ($t = 0, 1, 2$) economy with two banks.[10] At $t = 0$, the banks are identical. Each of them holds a unit portfolio of long-term assets, and finances the portfolio

[10] It should be emphasized that all results of the current paper can be easily generalized to a N-bank case.

with equity E, retail deposits F, and short-term wholesale debt $1 - E - F$. There are two groups of active players: banks' wholesale creditors and uninformed buyers of banks' assets. Both groups of players are risk neutral. We assume that retail deposits are fully insured so that depositors act only passively. Since their claims are risk free, the depositors will always hold their claims to maturity, and demand only a gross risk-free rate which we normalize to 1. It is assumed that the financial safety net is provided to banks free of charge. We consider banks as contractual arrangements among claim holders, designed to fulfil the function of liquidity and maturity transformation (Diamond and Dybvig (1983)). Therefore, banks in our model are passive, with given loan portfolios and liability structures.

Banks' wholesale debt is risky, demandable, and raised from a continuum of creditors. Provided that a bank does not fail, a wholesale debt contract promises a gross interest rate $r_D > 1$ at $t = 2$, and qr_D if a wholesale creditor withdraws early at $t = 1$. Here $q < 1$ reflects the penalty for the early withdrawal. A bank run occurs if a positive mass of wholesale creditors withdraw funds from their bank at $t = 1$. For the ease of future exposition, we denote by D_1 the total amount of debt a bank needs to repay at $t = 1$ if *all* wholesale creditors withdraw early, and by D_2 the total amount of debt a bank needs to repay at $t = 2$ if *no* wholesale creditor withdraws early.

$$D_1 \equiv (1 - E - F)qr_D$$

$$D_2 \equiv (1 - E - F)r_D + F$$

A bank's portfolio generates a random cash flow $\tilde{\theta}$ at $t = 2$. For simplicity, we assume that $\tilde{\theta}$ follows a uniform distribution on $\left[\underline{\theta}_s, \overline{\theta}\right]$, and the random cash flows of the two banks are independent and identically distributed. Subscript s denotes the realization of a systematic risk that affects both banks. There are two possible aggregate states, G and B (e.g., housing market boom or bust), and the two states occur with equal probability. With $\underline{\theta}_G > \underline{\theta}_B$, State G is more favorable than State B. Therefore, the value of a bank's assets is not only affected by its idiosyncratic risk $\tilde{\theta}$ but also by the aggregate state s. On the other hand, $\overline{\theta}$ is assumed to be the same across states. This reflects the fact that banks hold mostly debt claims whose highest payoffs are capped by their face values. We make the following four further parametric assumptions.

$$
\begin{aligned}
& D_2 > \underline{\theta}_s && (1)\\
& \frac{\theta_B + \overline{\theta}}{2} > D_2 && (2)\\
& F > D_1 && (3)\\
& q > \frac{1}{2} + \frac{\theta_B}{2D_2} && (4)
\end{aligned}
$$

Inequality (1) states that there is a positive probability of bankruptcy in each state. Inequality (2) states that, in the absence of bankruptcy costs, even if the realization of the aggregate state is unfavorable, the expected cash flow from a bank's assets is still greater than its debt obligations, so that bank lending is viable. Combined together, inequality (1) and (2) state that banking is efficient but risky. Inequality (3) suggests that a bank's retail debt exceeds its wholesale debt,[11] which is a realistic scenario and helps to simplify the analysis of bank run games.[12] Finally, inequality (4) suggests that the penalty on early withdrawal is only moderate,[13] which is in line with banks' role as liquidity providers (Diamond and Dybvig (1983)). While we do not endogenize banks' capital structure (therefore taking q, D_1, and D_2 as given), as long as the optimal capital structure satisfies the aforementioned conditions, all of our results will apply.

Banks' assets are long-term, taking two periods to mature. In particular, we assume that at $t = 1$ the assets cannot be physically liquidated. Therefore, if a wholesale run happens, to meet the liquidity demand, a bank has to financially liquidate its assets in a secondary market, and sell them to outside asset buyers. Because early liquidation is costly in this model, a bank will sell its assets if and only if it faces a bank run.

2.1 Secondary asset market

Buyers in the secondary asset market are uninformed: they observe neither the aggregate state s nor any bank's cash flow θ. Yet, they can observe the number of bank runs, and based on the observable outcome, form rational expectations about the quality of assets on sale. In this

[11] Note that for $q < 1$, inequality (3) also implies $D_2 > 2D_1$, because $D_2 = D_1/q + F > D_1 + F > 2D_1$.

[12] The condition is more than a technical assumption. It is realistic in the sense that despite of the rapid growth of wholesale funding, most commercial banks and bank holding companies are still financed more by retail deposits than wholesale debt.

[13] For example, when $\underline{\theta}_B = 0$, the condition states that $q > 1/2$. That is, by withdrawing early, a wholesale creditors will not lose more than a half of the face value of his claim.

two-bank setup, there are three distinct outcomes from the buyers' perspective, i.e., the number of bank runs $N = 0$, 1, or 2.

We assume the following sequential moves between asset buyers and wholesale creditors.[14] Asset buyers first post a price scheme $\mathbb{P} = (P_1 \ P_2)$, and offer to purchase bank assets on sale at price P_1 when the number of bank runs N equals 1, and at P_2 when $N = 2$. Having observed the price scheme, wholesale creditors play a bank run game, making their individual decisions simultaneously on whether to withdraw their funds early. In case any bank run happens, transactions take place at the offered price, and assets are transferred to buyers.

The price scheme $\mathbb{P}$ is complete in the sense that an asset price is specified for each distinct outcome of the bank run game where bank assets are on sale. Depending on the number of runs observed, the prices that buyers offer can differ. In fact, in the absence of commitment power, the asset buyers' decisions need to be time consistent so that they will not revoke their posted price after the outcomes of bank run games are revealed. As a result, the price P_1 and P_2 will have to reflect buyers' posterior beliefs on asset qualities. As buyers form different posterior beliefs upon observing different numbers of bank runs, their offered prices will vary with the number of runs.

The asset market is assumed to be perfectly competitive, and the buyers compete in the price schemes that they offer. In equilibrium, based on their posterior beliefs, the asset buyers should perceive themselves breaking even in expectation when purchasing bank assets at their posted prices. As the buyers make time-consistent decisions and do not revoke their offers, they must make no loss for any realized number of bank runs.

2.2 The bank run game

The demandable nature of wholesale debt allows creditors to withdraw their funds early, forcing a bank to liquidate its assets before maturity. When the assets are sold for less than their fundamental values, there will be a liquidation loss, or an asset fire sale. While creditors who withdraw early can avoid suffering from the fire sale, those who do not withdraw will lose if the bank fails. As a result, creditors' actions to withdraw display strategic complementarities, and it can be in the interest of all creditors to run on a bank that is otherwise solvent.

[14]It should be emphasized that the results of the model are robust to timing. If asset buyers and wholesale creditors move simultaneously, one can derive the same results based on the rational expectations of both parties.

A bank run game of complete information can have two strict equilibria: all creditors withdraw from the bank, and nobody withdraws. To refine the equilibria, we take the global-games approach pioneered by Carlsson and Van Damme (1993) and study games with incomplete information where common knowledge on θ does not exist among creditors. We assume that at the beginning of $t = 1$, both systematic risk (State s) and idiosyncratic risk (cash flow $\tilde{\theta}$) have been realized, but the information is not fully revealed to players. For a given bank, an individual creditor i privately observes a noisy signal $x_i = \theta + \epsilon_i$. Here, the noise ϵ_i is drawn from a uniform distribution with a support $[-\epsilon, \epsilon]$, where ϵ can be arbitrarily small. Based on their private signals, the creditors play a bank-run game with each other. Each of the creditors has two possible actions: to wait until maturity or to withdraw early, and follows a threshold strategy: to withdraw early if and only if their individual private signal is lower than a critical level $\hat{x}$. In this two-bank setup, we also assume that each creditor holds claims in both banks, and observes independent noisy signals for both banks' cash flows.[15]

The maturity mismatch between banks' liabilities and assets, together with potential asset fire sales, exposes banks to the risk of runs. In particular, a run and premature liquidation at $t = 1$ can cause the failure of a bank that is otherwise solvent at $t = 2$. In order to reassure its creditors not to withdraw early, a bank has to be more than merely solvent, and the market value of its equity should be able to absorb potential fire-sale losses. This implies a critical cash flow $\hat{\theta} > D_2$ for a bank to survive a run. The distance between $\hat{\theta}$ and D_2 provides a measure of financial fragility. Moreover, a lower asset price implies greater fire-sale losses and, therefore, a higher critical cash flow $\hat{\theta}$ for a bank to survive a run.

We assume that bankruptcy costs are sufficiently high such that once a bank fails, the wholesale creditors get zero payoff and only a senior deposit insurance company obtains the residual value.[16] Given this assumption, if a bank is to fail at $t = 1$, a wholesale creditor will receive zero payoff whether he withdraws early or not. In this case of indifference, we assume that the creditor will always withdraw. One justification is that wholesale creditors receive reputational benefits by running on a bank that is doomed to fail.[17]

[15] It is not uncommon for institutional investors to hold demandable debt claims in multiple banks. A similar assumption can be found in Goldstein and Pauzner (2004).

[16] It should be noted that this is off equilibrium in the model, because the wholesale debt is demandable and the wholesale creditors will withdraw early at $t = 1$, before the bank fails.

[17] The reputational benefit may come from the fact that the creditor makes a "right decision". More detailed discussion on this assumption can be found in Rochet and Vives (2004). The authors argue that the vast majority of wholesale deposits are held by collective investment funds, whose managers are compensated if they build a good reputation, and penalized otherwise.

2.3 Asymmetric information on cash flow θ

Intelligent asset buyers can solve the creditors' bank run game and thereby form rational beliefs on the qualities of assets on sale. In particular, they know that a bank will be forced into liquidation if and only if its cash flow is below $\hat{\theta}$. However, the lack of more detailed information makes solvent banks (those with $D_2 \leq \theta < \hat{\theta}$) indistinguishable from the insolvent ones (those with $\theta < D_2$). An equilibrium asset price will reflect only the average quality of assets on sale. Therefore, a bank with cash flow θ greater than the equilibrium price but less than $\hat{\theta}$ will experience an asset fire sale.

A lower asset price pushes $\hat{\theta}$, the critical cash flow necessary to avoid a run, upwards. Thus, there will be two-way feedback between asset fire sales and bank runs. When asset buyers offer a low price for a bank's assets, a run is triggered, which generates the pooling of assets, and thus fully justifies the low asset price offered in the first place. As a result, fire sales and bank runs can reinforce each other.

2.4 Belief updating on State s

While asset buyers hold a prior belief that State B and G occur with equal probability, after observing any bank runs, they update their beliefs according to Bayes' rule and consider State B to be more likely. The pessimistic belief updating can lead to financial contagion. In particular, a bank may face no runs if the other bank does not face a run, but will face a run if the other bank does. This defines financial contagion in our model.

In our model, financial contagion is self-fulfilling. When observing more bank runs, asset buyers infer State B to be increasingly likely, and reduce the asset price that they offer accordingly. The fear of increased liquidation losses makes wholesale creditors panic even more, and can lead to simultaneous bank runs in the first place.

2.5 Timing

The timing of the model is summarized in Figure 1. Events at $t = 1$ take place sequentially.

Figure 1: Timing of the game

t = 0	t = 1	t = 2
Banks are established, with their portfolios and liability structures as given.	1. s and $\tilde{\theta}$ are realized. 2. Asset buyers post a price scheme. 3. For each bank that they lend to, creditors receive private noisy signals about the bank's cash flow θ, and decide to run or not. 4. After observing the number of bank runs, buyers purchase assets at the quoted price.	1. Returns become public. 2. Remaining obligations are settled.

3 Mutually reinforcing bank runs and asset fire sales

Depending on the realization of $\tilde{\theta}$, the model can have two types of equilibria: one type with bank runs, and the other without. A market equilibrium with bank runs consists of two components. First, the bank run game features a threshold equilibrium. That is, when N runs happen and bank assets are sold for an equilibrium price P_N^*, a bank will experience a run if and only if its cash flow is lower than a unique threshold $\theta_N^* \equiv \hat{\theta}(P_N^*)$, $N \in \{1, 2\}$. Second, the competitive asset market is in a rational expectations equilibrium. That is, asset buyers form a rational belief about the quality of assets on sale based on the observed number of bank runs N. In particular, they anticipate $\theta < \hat{\theta}(P_N^*)$, and Batesian update their beliefs on State s. According to such posterior beliefs, asset buyers who purchase bank assets at an equilibrium price P_N^* should perceive themselves to break even in expectation. Moreover, the buyers should find themselves unable to profitably deviate from bidding P_N^*.

Definition. *Denote $\hat{\theta}(P_N)$ the threshold equilibrium of the bank run game for a given asset price P_N; and $P_N\left(\hat{\theta}\right)$ the price scheme by which asset buyers break even in expectation for a given threshold $\hat{\theta}$ and their rational beliefs about θ and s. An equilibrium associated with N bank runs, $N \in \{1, 2\}$, is defined by an equilibrium critical cash flow $\theta_N^* \equiv \hat{\theta}(P_N^*)$ and an equilibrium asset price $P_N^* \equiv P_N(\theta_N^*)$. The combination of θ_N^* and P_N^* is such that: for the N bank runs in the economy, (1) a successful run happens to a bank if and only if the bank's cash flow is lower than θ_N^*; (2) the competitive secondary asset market is in a rational expectations equilibrium: asset buyers form rational beliefs about State s and the quality of assets on sale, and based on their posterior beliefs, perceive themselves to make zero profit in expectation by* *purchasing bank assets for P_N^*. Furthermore, the buyers cannot make any profitable deviation by bidding any price other than P_N^**

It takes four steps to solve for a bank run equilibrium.

- First, defining $\underline{P} \equiv (\underline{\theta}_B + D_2)/2 \in (D_1, D_2)$, we show that an equilibrium asset price P_N^* cannot be lower than $\underline{P}$, nor higher than or equal to D_2 (subsection 3.1). This restricts the set of candidate equilibria and will facilitate the solution of bank run games.
- Second, solving the model using backward induction, we start with creditors who move last and solve the bank run game using the concept of global games. For a *given* asset price $P_N \in [\underline{P}, D_2)$, we derive a unique critical cash flow $\hat{\theta}(P_N)$, so that a bank run will happen if and only if the bank's cash flow $\theta < \hat{\theta}(P_N)$ (subsection 3.2).
- Third, we characterize asset buyers' posterior beliefs on asset qualities when N bank runs occur. In particular, they expect only those assets with quality $\theta < \hat{\theta}(P_N)$ to be on sale, and update their beliefs about State s using Bayes' rule. It should be emphasised that the buyers' rational beliefs are functions of asset prices that they offer (subsection 3.3).
- Finally, we solve for the equilibrium of the model by examining equilibrium asset prices. As asset buyers offer different prices given different numbers of bank runs, we solve for an equilibrium price P_N^* for each $N \in \{1, 2\}$. In a competitive equilibrium, P_N^* should be equal to the expected asset quality based on buyers' posterior beliefs upon observing N bank runs (subsection 3.3).

To illustrate the main intuition behind the feedback between bank runs and asset fire sales, we present in subsection 3.4 a simplified version of the model where there is only one state so that asset buyers do not update their beliefs on State s. This simplification allows us to derive a closed-form solution to our model, and is sufficient to generate some interesting results such as the unintended liquidity consequences of bank capital. The fully-fledged model with different states is analyzed in section 4.

3.1 Restricting the set of candidate equilibria

Since an equilibrium price cannot be negative, a candidate equilibrium price P_N^* must be in one of the three regions: $[0, \underline{P})$, $[\underline{P}, D_2)$, or $[D_2, +\infty)$. We will show that there exists a unique equilibrium price $P_N^* \in [\underline{P}, D_2)$ associated with N bank runs. And we start with showing that P_N^* cannot be lower than $\underline{P}$, nor can it be greater than or equal to D_2.

Suppose $P_N^* \geq D_2$. It follows that for any bank with $\theta \in [D_2, \overline{\theta}]$, it is suboptimal for its wholesale creditors to withdraw early. Because given $P_N^* \geq D_2$, an asset sale at $t = 1$ will not hurt the bank's capability to repay its liabilities at either $t = 1$ or $t = 2$. As a result, by running on the bank, a creditor will only incur the penalty for early withdrawal. This implies that whenever a run happens, the bank must be fundamentally insolvent with $\theta < D_2$. Therefore, the highest asset quality that buyers can expect is D_2, with the expected quality strictly lower than that. To avoid making losses, the price that the buyers pay must be lower than or equal to the expected quality, and therefore strictly lower than D_2, which contradicts the presumption $P_N^* \geq D_2$.

An equilibrium price P_N^* cannot be smaller than $\underline{P}$ either. To see so, note that when a bank is fundamentally insolvent with cash flow $\theta < D_2$, it is a dominant strategy for its wholesale creditors to run. Specifically, if $P_N^* \geq D_1$ so that the bank does not fail at $t = 1$, a creditor is better off to run and receive qr_D than to wait and receive 0. On the other hand, if $P_N^* < D_1$, a creditor will receive a zero payoff for his claim whether he runs or not, but can still obtain reputational benefits by running on a bank that is doomed to fail. This implies that runs must happen to those banks with $\theta < D_2$, and the expected quality of assets on sale is at least $(\underline{\theta}_B + D_2)/2 = \underline{P}$. Since asset buyers only break even in equilibrium, the price they offer must be greater than or equal to $\underline{P}$. We summarize these results in Lemma 1.

Lemma 1. *An equilibrium asset price cannot be greater than or equal to D_2, and cannot be smaller than $\underline{P}$.*

A corollary of $P_N \geq \underline{P}$ is that banks do not fail at $t = 1$. This is because $P_N \geq \underline{P} > D_1$ so that banks can always repay their $t = 1$ liabilities. Runs on the intermediate date, however, accelerate bank failures because fire-sale losses lead to a higher probability of $t = 2$ bankruptcy.

3.2 Threshold equilibrium for bank run games

We solve the model by backward induction, and start with the subgame of bank runs. We show that for a *given* price $P_N \in [\underline{P}, D_2)$ the bank run game has a unique threshold equilibrium characterized by a critical cash flow $\hat{\theta}(P_N)$. A successful bank run happens if and only if the bank's cash flow is lower than $\hat{\theta}(P_N)$.

To solve for the optimal strategy of creditors, we first derive their payoffs for action "wait" and "withdraw" as functions of the number of other creditors who withdraw early. Denote by

$L \in [0, 1]$ the fraction of creditors who withdraw from the bank at $t = 1$. A bank that faces a total withdrawal of LD_1 can meet the demand for liquidity with a partial liquidation by selling a fraction λ of its assets.[18]

$$\lambda = \frac{LD_1}{P_N} < 1 \tag{5}$$

After liquidating a fraction λ of its assets, the bank will fail at $t = 2$ if and only if the value of its remaining assets is lower than its remaining liabilities. That is,

$$(1 - \lambda)\theta < F + (1 - L)(1 - E - F)r_D. \tag{6}$$

Thus, a bank will fail at $t = 2$ if and only if the fraction of creditors' withdrawal exceeds a threshold L^c.

$$L > \frac{P_N[\theta - F - (1 - E - F)r_D)]}{(q\theta - P_N)(1 - E - F)r_D} = \frac{P_N(\theta - D_2)}{(\theta - P_N/q)\, D_1} \equiv L^c. \tag{7}$$

Such a $t = 2$ failure happens because the partial early liquidation incurs a fire-sale loss. When a sufficiently large number of creditors withdraw and the bank is forced to liquidate a significant share of assets prematurely, the remaining assets will not generate sufficient cash flows to meet the remaining liabilities. The creditors who withdraw early at $t = 1$ therefore can impose negative externalities on creditors who choose to wait.

Depending on the number of early withdrawals L, a creditor's payoffs of playing "withdraw" or "wait" are tabulated as follows.

	$L \in [0, L^c)$	$L \in [L^c, 1]$
withdraw	qr_D	qr_D
wait	r_D	0

Note that if a creditor withdraws, his payoff will always be $W_{run}(L) = qr_D$. Instead, if he waits, his payoff depends on the action of other creditors.

$$W_{wait}(L) = \begin{cases} r_D & L \in [0, L^c] \\ 0 & L \in [L^c, 1] \end{cases}$$

[18]Here $\lambda < 1$ is guaranteed by $P_N > D_1$. Note that three factors contribute to the amount of early liquidation: (i) a large number of early withdrawals, (ii) a low price for assets on sale, and (iii) a high level of wholesale debt.

Defining the difference between the creditor's payoffs of "withdraw" and "wait" as $DW(L) \equiv W_{wait}(L) - W_{run}(L)$, we have the following expression.

$$DW(L) = \begin{cases} (1-q)r_D & L \in [0, L^c] \\ -qr_D & L \in [L^c, 1] \end{cases}$$

The strategic complementarity is clear: when a sufficiently large number of other creditors choose to withdraw ($L > L^c$), a wholesale creditor receives better payoff from "withdraw" than from "wait". In fact, when there is complete information on θ, the bank run game has two equilibria in which either all creditors withdraw or all creditors wait. We refine the multiple equilibria using the technique of global games.

The analysis follows a standard global-games approach. We give here the outline of the proof, and interested readers can refer to Appendix A for full details. First, we establish the existence of a lower dominance region $[\underline{\theta}_s, \theta^L)$, where it is a dominant strategy for all wholesale creditors to withdraw early, independently of the private signals that they receive. Similarly, we show that, provided $\overline{\theta} > F/(1 - D_1/\underline{P})$, there exists an upper dominance region $\left[\theta^U(P_N), \overline{\theta}\right]$, where it is a dominant strategy for all creditors to wait.[19] For an intermediate $\theta \in \left[\theta^L, \theta^U(P_N)\right)$, a creditor's payoff depends on the actions of other creditors. So, as a second step, we characterize a creditor i's ex-post belief about the other creditors' actions. The creditor's belief is a conditional distribution of L based on his private signal $x_i = \theta + \epsilon_i$. The creditor then chooses his optimal action based on his ex-post belief and payoff function $DW(L)$. Finally, for the limiting case where the noise of the signal approaches zero, we obtain a unique threshold

$$\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N} \tag{8}$$

such that a successful bank run will happen if and only if the bank's cash flow $\theta < \hat{\theta}(P_N)$. The results are summarized in Proposition 1.

Proposition 1. *For a given secondary market asset price $P_N \in [\underline{P}, D_2)$, the bank run game has a unique threshold equilibrium: a successful run occurs to a bank if and only if the bank's cash flow falls below a critical level $\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N}$.*

Proof. See Appendix A. □

[19]One can derive θ^L explicitly as $\theta^L = D_2$ and show that θ^U has an upper bound $F/(1 - D_1/\underline{P})$.

Expression (8) establishes a one-to-one correspondence between a *given* asset price P_N and a corresponding critical cash flow $\hat{\theta}(P_N)$. Note that the critical cash flow $\hat{\theta}(P_N)$ decreases in P_N. A lower asset price makes successful bank runs more likely.

3.3 Asset market equilibrium

The uninformed asset buyers observe neither cash flow θ nor State s, but they can form rational beliefs about the quality of assets on sale. First, they anticipate the threshold equilibrium for the bank run game to be characterized by $\hat{\theta}(P_N)$. Therefore, when N bank runs happen, the buyers form a rational belief that only those assets of quality $\theta < \hat{\theta}(P_N)$ will be on sale. Second, the asset buyers also update their beliefs about State s using Bayes' rule. We denote by $\omega_N^G\left(\hat{\theta}(P_N)\right)$ the buyers' posterior belief that $s = G$ when the observed number of bank runs equals N, and by $\omega_N^B\left(\hat{\theta}(P_N)\right)$ their posterior belief for $s = B$. It should be emphasized that the posterior beliefs depend on buyers' offered price P_N.

Note that two factors contribute to asset fire sales. First, conditional on a bank run having happened, the cash flow of the bank must be lower than $\hat{\theta}(P_N)$. The buyers face an adversely selected pool of assets in the sense that only those banks with low cash flow will be forced into liquidation. Second, any observed bank runs also indicate that $s = B$ is more likely. This further reduces the expected quality of assets on sale, which in turn reduces asset buyers' willingness to pay.

When the asset market is perfectly competitive, an equilibrium asset price must satisfy two conditions. First, based on their rational expectations about θ and s, the buyers should make a zero expected profit by purchasing bank assets at the posted price. In other words, when there are N bank runs, an equilibrium asset price P_N^* must be equal to the expected asset quality.

$$P_N^* = E\left[\theta|\theta < \hat{\theta}(P_N^*), N\right] = \omega_N^B\left(\hat{\theta}(P_N^*)\right)\frac{\theta_B + \hat{\theta}(P_N^*)}{2} + \omega_N^G\left(\hat{\theta}(P_N^*)\right)\frac{\theta_G + \hat{\theta}(P_N^*)}{2} \tag{9}$$

Second, buyers should not be able to make any profitable deviation by unilaterally bidding a higher price. Therefore, their expected net payoff, $\Pi_N(P_N) \equiv E\left[\theta|\theta < \hat{\theta}(P_N), N\right] - P_N$, should not increase in their bid P_N.

The equilibrium has a fixed-point representation: P_N^* should be a fixed point for function $E\left[\theta|\theta < \hat{\theta}(P_N), N\right]$. We show that for *each* $N \in \{1, 2\}$, the fixed-point equilibrium exists and is unique. We also verify that the equilibrium is stable in the sense that buyers cannot profitably deviate by unilaterally bidding a higher price.

3.4 A baseline model

The feedback between a bank run and an asset fire sale can be examined without different aggregate states. To illustrate the main intuition, we analyze a baseline case of our model with $\underline{\theta}_B = \underline{\theta}_G = \underline{\theta}$. In this case, buyers do not update their beliefs about State s, so their posted price scheme will consist of only one unified price P. For this baseline model, we denote the market equilibrium by $\{\theta_e, P_e\}$, and obtain a closed-form solution.

As discussed, intelligent asset buyers can solve the subgame of bank runs and anticipate only those assets of quality $\theta < \hat{\theta}(P)$ to be on sale. When the asset market is in a competitive equilibrium, asset buyers who purchase banks' assets for price P should break even in expectation. Given their belief $\theta \sim U\left(\underline{\theta}, \hat{\theta}(P)\right)$, a candidate equilibrium price P_e must satisfy the following zero-profit condition.

$$P_e = \frac{\hat{\theta}(P_e) + \underline{\theta}}{2} \tag{10}$$

With $\hat{\theta}(P)$ derived in equation (8), we can write the condition explicitly

$$P_e = \frac{1}{2}\left(\frac{D_2 - D_1}{1 - qD_1/P_e} + \underline{\theta}\right), \tag{11}$$

and further rearrange it into a quadratic form

$$2P_e^2 - \Psi P_e + q\underline{\theta}D_1 = 0, \tag{12}$$

where $\Psi \equiv (D_2 - D_1) + 2qD_1 + \underline{\theta}$. As equation (12) has one and only one root in interval $(\underline{P}, D_2)$, we obtain the following closed-form solution for the equilibrium asset price.

$$P_e = \frac{\Psi + \sqrt{\Psi^2 - 8qD_1\underline{\theta}}}{4} \tag{13}$$

For P_e to be an equilibrium, asset buyers should not be able to make a profitable deviation by unilaterally bidding a higher price. That is, a buyer's expected payoff, $E\left[\theta|\theta < \hat{\theta}(P)\right] - P$, should not increase in P. In the baseline model, asset buyers' expected payoff takes the form

$\Pi(P) = \left(\hat{\theta}(P) + \underline{\theta}\right)/2 - P$, and it monotonically decreases in P.

$$\frac{d\Pi(P)}{dP} = \frac{1}{2}\frac{\partial\hat{\theta}(P)}{\partial P} - 1 < 0$$

By equation (11), we know that the equilibrium asset price is such that $\Pi(P_e) = 0$, so an asset buyer will earn negative profit by unilaterally bidding a price $P > P_e$. Intuitively, by bidding a higher price, a buyer decreases her expected payoff in two ways. First, a higher bid increases the cost for acquiring a piece of asset, which directly reduces her payoff. Second, a higher price P also alleviates the bank run risk, making fewer banks sell for liquidity reasons. As a result, the buyer faces a pool of assets with deteriorating quality where more banks are selling assets because of fundamental insolvency. This again reduces her expected payoff.

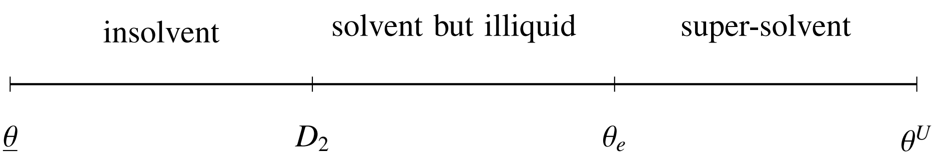
Having solved P_e, we can obtain the corresponding equilibrium critical cash flow $\theta_e \equiv \hat{\theta}(P_e)$ from expression (10). One can also verify that $\theta_e \in \left(\theta^L, \theta^U(P_e)\right)$.

$$\theta_e = \frac{\Psi + \sqrt{\Psi^2 - 8qD_1\underline{\theta}} - 2\underline{\theta}}{2} \tag{14}$$

The market equilibrium $\{\theta_e, P_e\}$ reflects asymmetric information on asset qualities. By offering P_e, an uninformed buyer makes a loss when $\theta < P_e$, and a profit when $\theta > P_e$. Furthermore, as a lower $\underline{\theta}$ aggravates the information asymmetry, it reduces the buyers' willingness to pay, and makes banks more likely to be illiquid. Mathematically, we have θ_e decreasing in $\underline{\theta}$.

Figure 2 illustrates the equilibrium funding liquidity risk. A bank with $\theta \in [D_2, \theta_e)$ may fully repay its debt obligations if no bank run happens, yet it will fail in equilibrium because of premature asset liquidation caused by the run of its wholesale creditors.

Figure 2: The unique equilibrium of the baseline model



Proposition 2. *The baseline model has a unique equilibrium, with equilibrium asset price* $P_e \in (\underline{P}, D_2)$ *and equilibrium critical cash flow* $\theta_e \in \left(\theta^L, \theta^U(P_e)\right)$ *specified in* (13) *and* (14)

respectively. A bank with cash flow $\theta \in [D_2, \theta_e)$ is solvent but illiquid: it will fail because of a wholesale debt run, even though its assets can generate a cash flow greater than its liabilities.

Proof. See Appendix B.1. □

3.5 Application I: capital and bank run risk

It is conventional wisdom that capital helps reduce the risk of bank runs. An application of the current framework, however, shows that the relationship is more subtle. We show that once asset prices are endogenous, capital also contributes to bank runs via stressed asset prices.

We model an increase of bank capital in its most simplistic form. We assume that a bank maintains its unit portfolio size, increasing its equity from E to $E + \Delta$, and at the same time decreasing its retail deposits from F to $F - \Delta$. In other words, an increase in capital reduces D_2 to $D_2 - \Delta$ but does not affect D_1.[20] We then examine how increasing capital affects banks' funding liquidity risks. To measure the risk, we follow Morris and Shin (2009) and define the equilibrium funding liquidity risk as $IL \equiv \hat{\theta}(P_e) - D_2$, with IL standing for illiquidity.[21]

Under exogenous asset prices, a natural corollary of Proposition 1 is that a higher capital always reduces funding liquidity risks, because the market value of capital serves as an extra buffer against fire-sale losses. The value of wholesale debt is better protected and wholesale creditors have less incentives to run, a channel that we call "buffer effect". Mathematically, recall that $\hat{\theta}(P) = \frac{D_2 - D_1}{1 - qD_1/P}$, we can write IL explicitly as

$$IL = \frac{D_2 - D_1}{1 - qD_1/P_e} - D_2. \tag{15}$$

With price P exogenous and not a function of Δ, it is straightforward to verify that increasing bank capital unambiguously reduces illiquidity.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} < 0 \tag{16}$$

With endogenous asset prices, the situation is more complicated. Once investors rationally update their beliefs about a bank's asset qualities, a higher capital level also contributes to

[20] An alternative assumption would involve reducing D_1, which directly decreases the amount of demandable short-term debt and therefore confounds the impact of changing capital on the risk of bank runs.

[21] Strictly speaking, the bank run risk should be measured as a probability $Prob\left(D_2 < \theta < \hat{\theta}(P_e)\right) = \frac{\hat{\theta}(P_e) - D_2}{\bar{\theta} - \underline{\theta}}$. We drop the the denominator because it is a constant and does not affect comparative statistics.

bank runs by reducing endogenous fire-sale prices. The intuition is as follows. In terms of inferring the realization of θ, a bank run presents more negative news when it happens to a well-capitalized bank than when it happens to a poorly-capitalized bank. Because a well-capitalized bank is able to sustain large losses, the fundamentals of the bank must be unusually poor for the run to happen. With such pessimistic inference about θ, buyers' willingness to pay for the bank's assets decreases with the observed capital level. Therefore, a change in bank capital affects illiquidity not only via D_2, but also via the endogenous asset price P_e.

$$\frac{\partial IL}{\partial \Delta} = \frac{\partial IL}{\partial D_2}\frac{\partial D_2}{\partial \Delta} + \frac{\partial IL}{\partial P_e}\frac{\partial P_e}{\partial \Delta} \tag{17}$$

The first term captures the traditional "buffer effect" as in the case where the asset price is exogenous. The second term captures a new channel that we want to emphasize: increasing capital also affects banks' funding liquidity risk via endogenous asset prices.

To see that higher capital leads to a lower secondary market asset price, one can simply take the first order derivative of the closed-form solution of P_e, which gives

$$\frac{\partial P_e}{\partial \Delta} = -\frac{1}{4} - \frac{1}{4}\frac{\Psi}{\sqrt{\Psi^2 - 8qD_1\underline{\theta}}} < 0.$$

Increasing capital decreases asset buyers' willingness to pay for a bank's assets on sale, which in turn makes creditors panic and bank runs more likely. This is captured by

$$\frac{\partial IL}{\partial P_e}\frac{\partial P_e}{\partial \Delta} > 0.$$

Hence, capital can contribute to funding liquidity risk by reducing endogenous asset prices, a mechanism we dub "inference effect". Comparing expression (16) with (17), it should be clear that with endogenous asset prices and the "inference effect", capital is less able to contain bank run risks as compared to the case where asset price is exogenous. Asset buyers' rational beliefs limit the role of capital in containing funding liquidity risk.

The overall impact of capital on funding liquidity risk depends on the relative strength of the "buffer effect" and the "inference effect". Using the closed form solution of P_e and θ_e, one can calculate the overall impact explicitly.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} + \frac{1}{4}\frac{q(D_2 - D_1)D_1}{(P_e - qD_1)^2}\left[1 + \frac{\Psi}{\sqrt{\Psi^2 - 8qD_1\underline{\theta}}}\right] \tag{18}$$

It can be shown that in the extreme case where $\underline{\theta} = 0$, $\partial IL/\partial\Delta = 0$ and increasing capital cannot reduce funding liquidity risk at all. Intuitively, a lower $\underline{\theta}$ reduces the expected quality of assets on sale, and therefore reduces buyers' willingness to pay. With

$$\frac{\partial}{\partial\underline{\theta}}\left(\frac{\partial P_e}{\partial\Delta}\right) > 0,$$

the price drop is most pronounced when $\underline{\theta} = 0$. In that case, the "inference effect" reaches its maximum and completely offsets the "buffer effect" of capital. We summarize the results in the following proposition.

Proposition 3. *In equilibrium, higher bank capital leads to a lower asset fire-sale price. Compared to the case where the price is exogenous, capital is less able to reduce the risk of bank runs. In the extreme case where $\underline{\theta} = 0$, higher capital does not reduce bank liquidity risk at all.*

Proof. See Appendix B.2 □

The result suggests that the design of prudential regulations has to take into account the responses of market participants. Compared to the situation where regulations are lax, market participants' interpretation of the same piece of negative news can be more pessimistic under stringent regulations. As a result, the effectiveness of stringent prudential regulations will be reduced, or even completely eliminated.

4 Self-fulfilling financial contagion

In this section, we extend the baseline model to include two aggregate states. Asset buyers will be able to update their beliefs about State s based on different numbers of bank runs. They perceive $s = B$ to be more likely when more bank runs are observed. In the absence of commitment power, the equilibrium prices that buyers offer must reflect their posterior beliefs, and therefore vary with the number of runs. We characterize market equilibrium associated with a single bank run and that associated with two bank runs, respectively. We show that for a given $N \in \{1, 2\}$, there exists a unique market equilibrium characterized by $\{P_N^*, \theta_N^*\}$ (section 4.1 and 4.2). We further establish that financial contagion can arise as a multiple-equilibria phenomenon, highlighting how pessmistic beliefs can drive financial fragility (section 4.3). Finally, we discuss how an asset purchase program committed by a regulator can improve financial stability over the market equilibria (section 4.4).

4.1 Market equilibrium with a single bank run

We start with characterizing the equilibrium associated with a single bank run. For a *given* asset price P_1 that corresponds to the single-bank-run outcome, the bank run game has a unique threshold equilibrium characterized by $\hat{\theta}(P_1)$. So asset buyers know that a bank run happens if and only if the bank's cash flow is lower than $\hat{\theta}(P_1)$, and update their beliefs about the aggregate state according to Bayes' rule. Recall that $\omega_1^s\left(\hat{\theta}(P_1)\right)$ denotes buyers' posterior belief for State s when they observe a single bank run.

$$
\begin{aligned}
\omega_1^B\left(\hat{\theta}(P_1)\right) \equiv Prob(s=B|N=1) &= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)} \\
&= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)} \\
\omega_1^G\left(\hat{\theta}(P_1)\right) \equiv Prob(s=G|N=1) &= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)} \\
&= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)}
\end{aligned}
$$

When the competitive asset market is in a rational expectations equilibrium, based on their posterior beliefs, asset buyers should expect to break even when purchasing bank assets for price P_1^*. Their *ex-post* zero-profit condition (9) can now be written as follows.

$$
P_1^* = E\left[\theta|\theta<\hat{\theta}(P_1^*), N=1\right] = \omega_1^B\left(\hat{\theta}(P_1^*)\right)\frac{\underline{\theta}_B+\hat{\theta}(P_1^*)}{2}+\omega_1^G\left(\hat{\theta}(P_1^*)\right)\frac{\underline{\theta}_G+\hat{\theta}(P_1^*)}{2} \tag{19}
$$

A candidate equilibrium price P_1^* should be a fixed point of function $E\left[\theta|\theta<\hat{\theta}(P_1^*), N=1\right]$. Alternatively, denoting by $\Pi_1(P_1)$ an asset buyer's *net* expected payoff when her offered price is P_1, we can re-write the zero-profit condition as in below.

$$
\Pi_1(P_1^*) \equiv \omega_1^B\left(\hat{\theta}(P_1^*)\right)\left(\frac{\underline{\theta}_B+\hat{\theta}(P_1^*)}{2}-P_1^*\right)+\omega_1^G\left(\hat{\theta}(P_1^*)\right)\left(\frac{\underline{\theta}_G+\hat{\theta}(P_1^*)}{2}-P_1^*\right)=0 \tag{20}
$$

Finding a fixed point P_1^* is equivalent to finding a solution for equation (20).

For P_1^* to be an equilibrium, an asset buyer must not profit by unilaterally raising her bid above P_1^*. In other words, function $\Pi_1(P_1)$ should not increase in P_1. Such monotonicity always holds, and the intuition is as follows. First of all, as discussed in section 3.4, increasing price raises the cost of acquiring bank assets and leads to an deteriorating quality in the asset pool. Second, when P_1 increases, the risk of bank run is mitigated, and a bank selling its assets is more likely to be fundamentally insolvent rather than facing a pure liquidity issue. For a *given* number of bank runs observed, this suggests that $s = B$ is more likely, $\partial \omega_1^B \left(\hat{\theta}(P_1)\right) / \partial P_1 > 0$, which further reduces the buyer's expected payoff.

Lemma 2. $\Pi_1(P_1)$ *monotonically decreases in* P_1*, so that given a single bank run is observed, a buyer's expected payoff monotonically decreases in her bid.*

Proof. See Appendix B.3. □

With extra complications introduced by the posterior beliefs on s, we can no longer obtain closed-form solutions for P_1^* and θ_1^*. Instead, we prove that there exists a unique $P_1^* \in (\underline{P}, D_2)$ that satisfies equation (19), and a corresponding $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$ by (8). The proof is based on the continuity of $\Pi_1(P_1)$. In particular, we show that $\Pi_1(P_1)$ is negative when $P_1 = \underline{P}$ and positive when $P_1 = D_2$. Furthermore, given the monotonicity of $\Pi_1(P_1)$, once an equilibrium exists, it is also unique. As a result, the market equilibrium associated with one bank run can be characterized by a unique pair $\{P_1^*, \theta_1^*\}$. The result is summarized in the proposition below.

Proposition 4. *There exist a unique equilibrium asset price* $P_1^* \in (\underline{P}, D_2)$ *and a unique equilibrium critical cash flow* $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$*, associated with one bank run. A bank with cash flow* $\theta \in \left[D_2, \theta_1^*\right)$ *is solvent but illiquid when one bank run happens.*

Proof. See Appendix B.4. □

4.2 Market equilibrium with two bank runs

Following the same approach as in the last subsection, we now characterize the market equilibrium associated with two bank runs. For a *given* asset price P_2 that corresponds to a two-bank-run outcome, a bank will fail because of a run if and only if its cash flow $\theta < \hat{\theta}(P_2)$. Again, we formulate asset buyers' posterior beliefs about State s according to Bayes' rule.

$$\omega_2^B \left(\hat{\theta}(P_2)\right) \equiv Prob(s = B|N = 2) = \frac{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2}{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}$$
$$\omega_2^G \left(\hat{\theta}(P_2)\right) \equiv Prob(s = G|N = 2) = \frac{\left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}$$

Based on the posterior beliefs, the asset buyers' break-even condition $P_2^* = E\left[\theta|\theta < \hat{\theta}(P_2^*), N = 2\right]$ can be written as follows.

$$P_2^* = \omega_2^B\left(\hat{\theta}(P_2^*)\right)\frac{\theta_B + \hat{\theta}(P_2^*)}{2} + \omega_2^G\left(\hat{\theta}(P_2^*)\right)\frac{\theta_G + \hat{\theta}(P_2^*)}{2} \tag{21}$$

Alternatively, denoting by $\Pi_2(P_2)$ an asset buyer's *net* expected payoff when her offered price is P_2, we can re-write the zero-profit condition as in below.

$$\Pi_2(P_2^*) \equiv \omega_2^B\left(\hat{\theta}(P_2^*)\right)\left(\frac{\theta_B + \hat{\theta}(P_2^*)}{2} - P_2^*\right) + \omega_2^G\left(\hat{\theta}(P_2^*)\right)\left(\frac{\theta_G + \hat{\theta}(P_2^*)}{2} - P_2^*\right) = 0 \tag{22}$$

Lemma 3 shows that buyers' expected payoff monotonically decreases in P_2, so that they have no profitable deviation. Thus, any solution to equation (22) is indeed a market equilibrium.

Lemma 3. $\Pi_2(P_2)$ *monotonically decreases in* P_2*, so that given two bank runs are observed, a buyer's expected payoff monotonically decreases in her bid.*

Proof. See Appendix B.5. □

To prove the existence of and uniqueness of the equilibrium, we again use the monotonicity and continuity of $\Pi_2(P_2)$. We show that $\Pi_2(P_2)$ is negative at $\underline{P}$ and positive at D_2, so that the market equilibrium associated with two bank runs can be characterized by a unique pair $\{P_2^*, \theta_2^*\}$. The result is summarized in Proposition 5.

Proposition 5. *There exist a unique equilibrium asset price* $P_2^* \in (\underline{P}, D_2)$ *and a unique equilibrium critical cash flow* $\theta_2^* \in (\theta^L, \theta^U(P_2^*))$*, associated with two bank run. A bank with cash flow* $\theta \in [D_2, \theta_2^*)$ *is solvent but illiquid when two bank run happens.*

Proof. See Appendix B.6. □

4.3 Financial contagion and multiple equilibria

$\theta_2^* > \theta_1^*$ would imply potential contagion. In particular, when a bank's cash flow lies between θ_1^* and θ_2^*, the bank will face no run if the other bank does not face a run, and will fail in a wholesale run if the other bank does. We prove with Lemma 4 that $\theta_2^* > \theta_1^*$ is indeed the case. Intuitively, asset buyers form more pessimistic beliefs about State s when they observe more bank runs. Their willingness to pay for banks' assets decreases because banks' expected asset qualities are lower in State B. This in turn reduces the equilibrium asset price and pushes upwards the equilibrium critical cash flow that a bank has to meet to survive a run.

Lemma 4. *When more runs are observed, the equilibrium asset price is lower, $P_2^* < P_1^*$, and the risk of bank runs is higher, $\theta_2^* > \theta_1^*$.*

Proof. See Appendix B.7. □

Financial contagion emerges as a multiple-equilibrium phenomenon in the current model. In fact, when a bank's cash flow $\theta \in [\theta_1^*, \theta_2^*)$ and the other bank's cash flow $\theta < \theta_2^*$, the equilibrium number of bank runs N depends on creditors' beliefs about each others' strategies. Notice that only two threshold strategies can be be rationalized as part of a market equilibrium, i.e., an optimistic threshold strategy, "to run if and only if $x < \theta_1^*$", and a pessimistic threshold strategy, "to run if and only if $x < \theta_2^*$". As a result, we can focus on these two threshold strategies only. We show that financial contagion can happen purely because of creditors' pessimistic beliefs.

For the ease of exposition, we label the two banks as i and j, and discuss the following two cases respectively. (1) Bank i has a cash flow $\theta \in [\theta_1^*, \theta_2^*)$, and Bank j has a cash flow $\theta < \theta_1^*$.[22] (2) Bank i and j both have cash flows between θ_1^* and θ_2^*.

In the first case, the equilibrium number of bank runs can be either 1 or 2, depending on creditors' belief about each others' strategies. With a cash flow $\theta < \theta_1^*$, Bank j will fail in a run whether creditors follow the optimistic or pessimistic strategy. Therefore, there will be at least one bank run in the economy. Whether Bank i will have a run, however, depends on creditors' beliefs. If creditors believe that a positive mass among them follow the pessimistic strategy, they will expect a run on Bank i and an asset price P_2^*, so that it is optimal to join the run. As a result, all creditors withdrawing early from Bank i can emerge as an equilibrium. By contrast, if all creditors believe that none of them follow the pessimistic strategy, they would expect the asset price to be P_1^*, and only Bank j to fail, which justifies their optimistic belief/strategy in the first place.

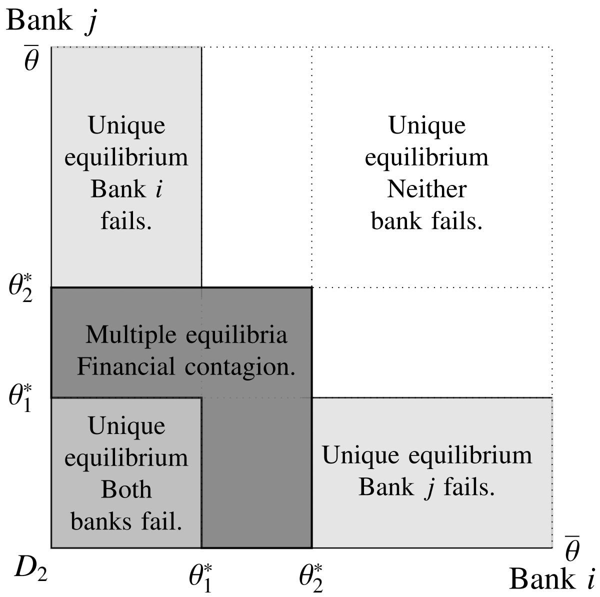
In the second case, the equilibrium number of bank runs can be either 0 or 2, depending again on creditors' beliefs. If all creditors believe that none of them follow the pessimistic strategy, no run will happen, because both banks' cash flows are higher than θ_1^*. Therefore, $N = 0$ can be an equilibrium outcome. In contrast, if a creditor believes that a positive mass among them follow the pessimistic strategy, he will expect two bank runs and assets sold for

[22] The symmetric case where Bank i has a cash flow $\theta < \theta_1^*$, and Bank j has $\theta \in [\theta_1^*, \theta_2^*)$ can be analyzed with the same reasoning.

price P_2^*, so that it is optimal for him to join the run. Therefore, $N = 2$ can emerge as an equilibrium outcome. The creditor's belief must be that a positive mass of creditors will run on each of the two banks. Because if the pessimistic creditors are only present in one bank, those creditors' strategy cannot be rationalized. Therefore, $N = 1$ cannot be an equilibrium outcome.

In sum, multiple equilibria can emerge when one bank's cash flow is in $[\theta_1^*, \theta_2^*)$ and the other bank's cash flow is below θ_2^*. The contagion is self-fulfilling and fuelled by creditors' beliefs. In Figure 3, we plot the possible equilibrium outcomes for different combinations of bank cash flows, and summarize the results in Proposition 6.

Figure 3: Equilibrium of the fully-fledged model



Proposition 6. *When one bank's cash flow belongs to* $[\theta_1^*, \theta_2^*)$ *and the other bank's cash flow is lower than* θ_2^**, multiple market equilibria exist, and financial contagion can happen because of creditors' pessimistic beliefs.*

4.4 Application II: asset purchase programs

We show in this section that a regulator with commitment power can promote financial stability even if she is not better informed than the asset buyers. The welfare-improving policy intervention that we propose resembles asset purchase programs such as Term Asset-Backed Securities Loan Facility (TALF).

We consider the following policy intervention: the regulator makes a promise to purchase bank assets at a unified price P_A in case any bank run happens. In particular, the price P_A does not depend on the number of bank runs in the economy. The regulator is assumed to have full commitment power and will not revoke his offer after having observed the actual number of runs. Under the policy intervention, the model has a revised timeline as depicted in Figure 4.

Figure 4: Timing of the asset purchase program

t = 0	t = 0.5	t = 1	t = 2
Banks are established, with their portfolios and liability structures as given.	The regulator makes a promise to buy assets at a unified price P_A, in case any bank run happens.	1. s and $\tilde{\theta}$ are realized. 2. Asset buyers post a price scheme. 3. For each bank that they lend to, creditors receive private noisy signals about the bank's cash flow θ, and decide to run or not. 4. Assets are sold to the party that offers the highest price.	1. Returns are realized. 2. Remaining obligations are settled

The regulator is risk-neutral, and is subject to an ex-ante budget constraint: she should not make any loss in expectation. To maximize social welfare, she will choose an optimal price P_A^* so that she only breaks even. This is because any lower price that leads to a positive expected profit will come at the cost of letting more solvent banks fail in runs.

The regulator is different from the typical asset buyers in the market because she holds full commitment power. In particular, she does not require to break even for each observed number of bank runs, but only to break even from an ex-ante perspective. Such commitment power allows the regulator to disregard new information such as the number of bank runs, and can therefore avoid the vicious cycle fuelled by pessimistic belief updating.

We now derive the ex-ante break-even price P_A^*. As a first step, we solve for price P_A^* under the assumption that banks will sell their assets to the regulator instead of to other asset buyers in the secondary market. When wholesale creditors expect bank assets to be sold at price P_A^*, we know from Section 3.3 that the critical cash flow of the bank run game is $\hat{\theta}(P_A^*)$. So the regulator understands that only those assets with $\theta < \hat{\theta}(P_A^*)$ will be on sale, with $\hat{\theta}(P_A^*)$ again defined by expression (8).

$$\hat{\theta}(P_A^*) = \frac{D_2 - D_1}{1 - qD_1/P_A^*} \tag{23}$$

As the regulator commits to price P_A^* before observing any bank runs, she holds the prior belief $Prob(s = G) = Prob(s = B) = 1/2$. From this ex-ante perspective, the regulator's break-even condition can be written as follows.

$$P_A^* = \frac{1}{2}\frac{\underline{\theta}_B + \hat{\theta}(P_A^*)}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \hat{\theta}(P_A^*)}{2} \tag{24}$$

Using expression (23), we can rewrite the ex-ante break-even condition (24) into a quadratic function of P_A^*, which has the following root between $\underline{P}$ and D_2.

$$P_A^* = \frac{\overline{\Psi} + \sqrt{\overline{\Psi}^2 - 8qD_1E(\underline{\theta})}}{4} \tag{25}$$

Here $E(\underline{\theta}) \equiv (\underline{\theta}_B + \underline{\theta}_G)/2$, and $\overline{\Psi} \equiv (D_2 - D_1) + 2qD_1 + E(\underline{\theta})$. Having obtained P_A^*, we can derive the corresponding $\hat{\theta}(P_A^*)$ using equation (23). Following the same procedure as in the proof of Proposition 2, we can show that $\hat{\theta}(P_A^*) \in (\theta^L, \theta^U(P_A^*))$.

Lemma 5. *If the regulator purchases banks' assets for a pre-committed price, she can break even from an ex-ante perspective by offering P_A^* as specified in equation (25).*

Proof. See Appendix B.8. □

We show in Proposition 7 that P_A^* is higher than what market offers (i.e., $P_A^* > P_1^* > P_2^*$), so that banks will indeed sell their assets to the regulator. As $\hat{\theta}(P)$ decreases with P, the policy intervention improves financial stability as compared to the market equilibria. In particular, the asset purchase committed by the regulator reduces (though does not eliminate) the risk of bank runs, and completely rules out financial contagion.

Proposition 7. *The regulator's ex-ante break-even price P_A^* is higher than the prices in market equilibria, so that banks will sell their assets to the regulator when they face runs. With $P_A^* > P_1^* > P_2^*$, and $\hat{\theta}(P_A^*) < \hat{\theta}(P_1^*) < \hat{\theta}(P_2^*)$, the regulator reduces the risk of bank runs and eliminates financial contagion.*

Proof. See Appendix B.9. □

With her commitment power, the regulator can disregard the outcome of bank run games and stick to a unified asset price. This allows the regulator to avoid the vicious cycle between bank runs and asset fire sales that is fuelled by pessimistic beliefs in market. As the regulator only needs to break even ex ante given her prior belief about State s, she can use the profit from State G to compensate the loss in State B.

Typical buyers in the market, on the other hand, are unable to do so. Without the commitment power, they must not make expected losses given any realized number of bank runs. In other words, they are constrained by ex-post break-even conditions. In fact, if an asset buyer offers the same price P_A^*, she will revoke the offer when a bank run actually happens, because in that case she will form a posterior belief that $s = B$ is more likely and will no longer consider herself breaking even by purchasing the bank assets at P_A^*. To break even from this ex-post perspective, the asset buyer has to lower her offered price, so as to decrease the loss from purchasing assets with $\theta \in [\underline{\theta}_s, P_N^*)$, and to increase the profit from purchasing assets with $\theta \in \left[P_N^*, \hat{\theta}(P_N^*)\right)$. The lack of commitment power therefore leads to a lower asset price, which in turn results in more bank runs, and justifies the pessimistic belief in the first place.

This result naturally relates to Lender of Last Resort (LOLR) policies. The effectiveness of such policies has long been debated, because a lender of last resort may not be able to tell whether a troubled bank is insolvent or illiquid. As a result, it is argued that, blinded intervention will compromise market discipline, whereas taking no action runs the risk of letting solvent banks fail. The current model, however, shows that interventions such as asset purchase programs can still improve financial stability without demanding information on individual banks' financial health. Even if such interventions are not perfect—banks with $\theta \in \left(D_2, \hat{\theta}(P_A)\right)$ still fail because of illiquidity—our model shows that the dilemma faced by a Lender of Last Resort is not as stark as one might think.

5 Further policy discussion

The model is rich enough for further policy analyses. We present here one more policy implication for regulatory disclosure.[23] We consider a scenario where a regulator has superior information about the aggregate state s and can credibly disclose the information to market

[23] In reality, examples of the regulatory disclosure include communicating stress testing parameters to the public or making announcements of the size of assistance programs.

participants. We analyze the cost and benefit of such regulatory disclosure, and compare it with asset purchase programs in terms of promoting financial stability.

5.1 Trade-offs for regulatory disclosure

To concentrate on the effects of disclosure, we focus on a simplistic case where the regulator observes State s perfectly. Once the regulator decides to disclose the information, the realization of s will be released before market trading. We also assume that the regulator can commit to truthful revelations by legislation. The information set of asset buyers changes correspondingly. Instead of updating beliefs about State s based on the number of bank runs, buyers now learn the state with certainty. Therefore, asset prices can be conditional on the true state that the regulator discloses.

The regulatory disclosure eliminates alternative beliefs as a source of multiple equilibria. Instead of two rational expectations equilibria depending on buyers' beliefs, there will be a unique equilibrium for each disclosed (realized) state. Denote by $\{\theta_G^*, P_G^*\}$ and $\{\theta_B^*, P_B^*\}$ the equilibrium critical cash flows and asset prices for State G and B respectively. Lemma 6 demonstrates the effect of regulatory disclosure on banks' illiquidity.

Lemma 6. *When the aggregate State s is disclosed to market participants, there exists a unique market equilibrium $\{\theta_s^*, P_s^*\}$ associated with each realized State $s \in \{G, B\}$. In State s, a bank with cash flow $\theta \in [D_2, \theta_s^*)$ will fail because of illiquidity. Such regulatory disclosure eliminates the multiple equilibria caused by the asset buyers' beliefs about the aggregate state.*

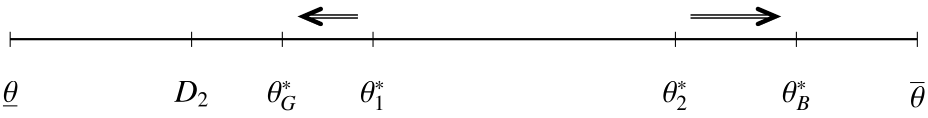
Proof. See Appendix B.10 □

Intuitively, we have $\theta_B^* > \theta_2^*$ and $\theta_G^* < \theta_1^*$. Here $\theta_B^* > \theta_2^*$ because even when observing two bank runs, buyers cannot exclude the possibility of $s = G$. But to the extent that a regulatory disclosure $s = B$ is accurate, buyers will bid according to $s = B$ with certainty. Similarly, making a favorable disclosure $s = G$ further reassures market participants and can save banks from illiquidity. The result is summarized in Proposition 8 and is illustrated by Figure 5.

Proposition 8. *With $\theta_G^* < \theta_1^*$ and $\theta_B^* > \theta_2^*$, the regulatory disclosure reduces illiquidity if $s = G$ but increases it if $s = B$.*

Proof. See Appendix B.11. □

Figure 5: Equilibrium under regulatory disclosure



The disclosed information, when favorable, boosts asset prices and saves banks with $\theta \in (\theta_G^*, \theta_1^*)$ from illiquidity. Acknowledging a bad state, on the other hand, exacerbates liquidity problems: solvent banks are more likely to suffer from illiquidity when market participants are aware of the downside risk, and banks with $\theta \in (\theta_2^*, \theta_B^*)$ will fail because of runs. Therefore, in determining whether to disclose information to the public, the regulator faces a trade-off: if the state is good, the reassuring disclosure can calm down the market and save banks from illiquidity; but if the state is unfavorable, acknowledging a crisis will result in even more runs by pushing asset prices further down. Arguably, when the social cost of bank failure is greater in State B, it can be suboptimal for regulators to commit to disclosing information.

5.2 Regulatory disclosure vs. Asset purchase

Now we run a horse race between different policy interventions, examining whether regulatory disclosures can outperform an asset purchase program as modelled in section 4.4. The regulator is assumed to choose between the two policy interventions before State s realizes. We show that an asset purchase program, which does not require superior information on the aggregate state, can actually achieve a higher level of financial stability.

Note first that asset purchase programs and regulatory disclosures are mutually exclusive in the current model. Once it has been credibly communicated that $s = G$, the equilibrium asset price will be P_G^*, which is higher than the break-even price P_A^* in asset purchase programs. The reason is that asset buyers are most optimistic when they learn the state is good with certainty. Suppose that an asset purchase program and regulatory disclosure coexist, the regulator will not acquire any assets in the good state unless her pre-committed price is higher than P_G^*. Such a price, however, implies making losses from an ex-ante perspective.

We then compare the asset purchase program and the regulatory disclosure. For simplicity, we concentrate on the social cost of bank failures. We denote by C the social cost associated with a bank failure and assume it to be independent of the number of bank failures and State s. We denote by SC_{AP} and SC_{RD} the expected social costs associated with the asset purchase program and the regulatory disclosure, respectively. Recall that $\hat{\theta}_A^*$ is the critical cash flow for the regulator's ex-ante break-even price P_A^*. SC_{AP} can be formulated as follows.

$$SC_{AP} = \frac{1}{2}\left[\frac{\theta_A^* - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} \cdot \frac{\overline{\theta} - \theta_A^*}{\overline{\theta} - \underline{\theta}_B} 2C + \left(\frac{\theta_A^* - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B}\right)^2 2C\right] + \frac{1}{2}\left[\frac{\theta_A^* - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} \cdot \frac{\overline{\theta} - \theta_A^*}{\overline{\theta} - \underline{\theta}_G} 2C + \left(\frac{\theta_A^* - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G}\right)^2 2C\right]$$

Here the first (second) square bracket corresponds to the total expected cost of bank failures in State B (State G), where there can be one or two bank failures. The expression can be further simplified as follows.

$$SC_{AP} = \frac{\theta_A^* - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} C + \frac{\theta_A^* - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} C \tag{26}$$

Similarly, the expected social cost associated with the regulatory disclosure can be written as

$$SC_{RD} = \frac{\theta_B^* - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} C + \frac{\theta_G^* - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} C. \tag{27}$$

We show with Corollary 1 that the social cost associated with asset purchase programs is lower if and only if $\overline{\theta} > \frac{\underline{\theta}_G \theta_B^* + \underline{\theta}_B \theta_G^* - (\underline{\theta}_B + \underline{\theta}_G)\theta_A^*}{\theta_B^* + \theta_G^* - 2\theta_A^*}$. This result stems from the fact that the critical cash flow $\hat{\theta}(P)$ is decreasing and convex in P, so that $\theta_A^* < (\underline{\theta}_B + \underline{\theta}_G)/2$.

Corollary 1. *The expected social cost due to bank failures is lower under asset purchase programs than under regulatory disclosures if and only if* $\overline{\theta} > \frac{\underline{\theta}_G \theta_B^* + \underline{\theta}_B \theta_G^* - (\underline{\theta}_B + \underline{\theta}_G)\theta_A^*}{\theta_B^* + \theta_G^* - 2\theta_A^*}$.

Proof. See Appendix B.12. □

6 Concluding remarks

In this paper, we have investigated the relationship between bank runs and asset fire sales. We have presented a model where fire sales and bank runs are driven by the lack of information and endogenously determined in a rational expectations equilibrium. We have also extended the model to incorporate financial contagion when there is a common risk exposure. We can draw several conclusions from our analysis. First, fire sales and bank runs and mutually reinforcing: when creditors anticipate low prices for a bank's assets, a run will be triggered, which generates asset fire sales and the corresponding collapse in prices, thus fully justifying creditors' strategies. Second, as one bank fails, asset buyers lower their expectations of the common risk factor and perceive banks' assets to be less valuable: the declining asset prices will precipitate contagious runs at all other banks.

The model has derived three policy implications regarding bank capital, asset purchase programs and regulatory disclosure. First, contrary to the conventional wisdom, we have shown that bank capital holding can have unintended consequences for funding liquidity, because a run on a well-capitalized bank signals unusually high risk and exacerbates asset fire sales, which in turn makes the run more likely. At the extreme, the model predicts that capital cannot reduce bank funding liquidity risk at all. Second, we have demonstrated that a regulator can break down the vicious circle between asset fire sales and bank runs by committing to purchase banks' assets at a predetermined price. By implementing such a pre-committed asset purchase program, regulators can promote financial stability and still break even from ex-ante perspective, even with no better information than other market participants. Finally, we have shown that regulatory disclosure is a double-edged sword. It saves banks from illiquidity when the disclosure is favorable. But, it amplifies funding liquidity risk and financial contagion when the disclosure worsens market beliefs.

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Appendix A Bank run game for $\underline{P} \leq P_N < D_2$

In this appendix, we solve creditors' bank run game for a *given* asset price $P_N \in [\underline{P}, D_2)$. We first establish the existence of a lower (upper) dominance region, where it is a dominant strategy for creditors to withdraw (wait) (Appendix A.1). Whereas outside of the dominance regions, a creditor will choose his action according to his beliefs about other creditors' actions. We characterize such beliefs in Appendix A.2. Finally, we prove that an equilibrium with a threshold strategy exists and is unique (Appendix A.3).

Appendix A.1 Lower and upper dominance regions

First, denoting the lower dominance region by $[\underline{\theta}, \theta^L)$, we prove its existence by construction. By definition, when a bank's cash flow belongs to the region, a creditor would be better off to withdraw even if all other creditors wait. This is the case if and only if inequality (6) holds for $L = 0$ so that $\theta < F + (1 - E - F)r_D = D_2$. In other words, the bank is fundamentally insolvent and will fail at $t = 2$ even if no premature liquidation takes place at $t = 1$. In this case, a creditor, if chooses to wait, will receive a zero payoff because of the bank failure, but will receive qr_D if he withdraws early. Therefore, if the creditor is sure that $\theta < D_2$ given his signal, his best action is to withdraw, independently of his belief about the other creditors' actions. As a result, we have the dominance region $[\underline{\theta}, \theta^L)$ with $\theta^L = D_2$.

Second, denoting $\theta^U(P_N) \equiv F/\left(1 - D_1/P_N\right)$ for a *given* asset price P_N, we show that the upper dominance region is $\left[\theta^U(P_N), \overline{\theta}\right]$. To see that it is a creditor's dominant strategy to wait, suppose that all other creditors withdraw early ($L = 1$) and the bank will survive if and only if its cash flow $\theta \geq \theta^U(P_N)$. In this case, if the creditor withdraws, he will receive qr_D; whereas if he waits, he will receive r_D. Again, when the creditor's signal is accurate, he will be sure that the bank's cash flow is in the upper dominance region, and will choose to wait independently of his belief about other creditors' actions. Note that $\theta^U(P_N)$ decreases in P_N, and we know that an equilibrium cannot be lower than $\underline{P}$. Therefore, $\theta^U(P_N)$ has an upper bound $F/\left(1 - D_1/\underline{P}\right)$. Provided that $\overline{\theta}$ is sufficiently high, $\overline{\theta} > F/\left(1 - D_1/\underline{P}\right)$, the upper dominance regions is guaranteed to be non-empty.

Finally, as we will verify later, $\theta^U > \theta^L$ will be automatically satisfied for equilibrium asset prices.

Appendix A.2 Creditors' beliefs outside the dominance regions

When the bank's realized cash flow is in the intermediate region $\left[\theta^L, \theta^U(P_N)\right)$, a creditor's optimal action depends on his beliefs about other creditors' actions. In this subsection, we characterize such beliefs. We assume that creditors act according to a threshold strategy. That is, creditors choose a threshold signal $\hat{x}$, and a creditor i withdraws at $t = 1$ if his signal $x_i < \hat{x}$, and waits until $t = 2$ if $x_i > \hat{x}$.

Note first that the fraction of creditors who withdraw early is a function of a bank's realized cash flow θ and the threshold signal $\hat{x}$, because an individual creditor's decision to withdraw or not depends both on the realization of the cash flow and on the strategy of other players. We denote this function by $L = L(\theta, \hat{x})$ and assume that creditors' signals are sufficiently accurate in the sense that the noise ϵ_i is distributed on an arbitrarily small interval $(-\epsilon, \epsilon)$, with $\epsilon \to 0$. In this limiting case, the threshold signal $\hat{x}$ implies a threshold cash flow $\hat{\theta}$. A creditor i will withdraw at $t = 1$ if $x_i < \hat{\theta}$, and will wait till $t = 2$ if $x_i > \hat{\theta}$. The fraction of early withdrawals can be rewritten as $L(\theta, \hat{\theta})$ accordingly.

Now, we determine the functional form of $L(\theta, \hat{\theta})$. For a realized θ, we have three cases. (1) When $\theta + \epsilon < \hat{\theta}$, even the highest possible signal is below the threshold $\hat{\theta}$. By the definition of the threshold strategy, all creditors will withdraw at $t = 1$ and $L(\theta, \hat{\theta}) = 1$. (2) When $\theta - \epsilon > \hat{\theta}$, even the lowest possible signal exceeds the threshold $\hat{\theta}$. All creditors will wait till $t = 2$ and $L(\theta, \hat{\theta}) = 0$. (3) When θ falls into the intermediate range $\left[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon\right]$, the fraction of creditors who withdraw at $t = 1$ is as follows.

$$L(\theta, \hat{\theta}) = Prob\left(x_i < \hat{\theta} | \theta\right) = Prob\left(\epsilon_i < \hat{\theta} - \theta | \theta\right) = \frac{\hat{\theta} - \theta - (-\epsilon)}{2\epsilon} = \frac{\hat{\theta} - \theta + \epsilon}{2\epsilon} \tag{A.28}$$

The realized fundamental θ looks uncertain from the perspective of creditors, because they receive only noisy signals of it. With the noise ϵ_i uniformly distributed on $[-\epsilon, \epsilon]$, a creditor who receives a signal x_i holds a posterior belief that θ follows a uniform distribution on $[x_i - \epsilon, x_i + \epsilon]$. Given that the fraction of creditors who withdraw is a function of the fundamental, each creditor forms a posterior belief about the fraction L based on his private signal.

We now derive those posterior beliefs. To begin with, we show that L is uniformly distributed on $[0, 1]$ from the perspective of a marginal creditor who happens to observe $x_i = \hat{\theta}$.

$$Prob\left(L(\theta, \hat{\theta}) \leq \hat{L} \middle| x_i = \hat{\theta}\right) = Prob\left(\frac{\hat{\theta} - \theta + \epsilon}{2\epsilon} \leq \hat{L}\right) = Prob\left(\theta \geq \hat{\theta} + \epsilon - 2\epsilon\hat{L}\right).$$

We also know that, conditional on $x_i = \hat{\theta}$, the marginal creditor has a posterior belief that θ is uniformly distributed on $[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]$, which implies $Prob\left(L(\theta, \hat{\theta}) \leq \hat{L} | x_i = \hat{\theta}\right) = \hat{L}$. Therefore, the marginal creditor holds a posterior belief that $L \sim U(0, 1)$.

Starting with case $x_i > \hat{\theta}$, we now analyze creditors who receive signals other than $\hat{\theta}$. Recall that a creditor who receives a signal x_i holds a posterior belief $\theta \sim U(x_i - \epsilon, x_i + \epsilon)$. For $x_i > \hat{\theta}$, the upper bound of the support is greater than $\hat{\theta} + \epsilon$, and we know that if $\theta > \hat{\theta} + \epsilon$, all creditors will wait. In fact, we can divide the support into two segments: $\left[x_i - \epsilon, \hat{\theta} + \epsilon\right]$ and $\left(\hat{\theta} + \epsilon, x_i + \epsilon\right]$, with the second segment associated with $L = 0$. A creditor i who receives $x_i > \hat{\theta}$, therefore, will perceive a positive probability mass for $L = 0$. On the other hand, the posterior belief of θ continues to have a uniform distribution on $\left[x_i - \epsilon, \hat{\theta} + \epsilon\right]$. To see so, note that $\left[x_i - \epsilon, \hat{\theta} + \epsilon\right] \subset \left[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon\right]$ given $x_i > \hat{\theta}$, so that $L(\theta, \hat{\theta})$ follows expression (A.28). Therefore, for $\theta \in \left[x_i - \epsilon, \hat{\theta} + \epsilon\right]$, we can derive the posterior belief on L as follows.

$$Prob\left(L(\theta, \hat{\theta}) \leq \hat{L} | x_i, x_i > \hat{\theta}\right) = Prob\left(\frac{\hat{\theta} - \theta + \epsilon}{2\epsilon} \leq \hat{L}\right) = Prob\left(\theta \geq \hat{\theta} + \epsilon - 2\epsilon\hat{L}\right)$$

Because the creditor perceives $\theta \sim U(x_i - \epsilon, \hat{\theta} + \epsilon)$, the probability above can be calculated as $2\epsilon\hat{L} / \left[2\epsilon - (x_i - \hat{\theta})\right]$, which implies that L follows a uniform distribution on $\left[0, 1 - \left(x_i - \hat{\theta}\right)/2\epsilon\right]$. Notice that the density function on this interval is 1. Thus, the probability uniformly distributed on this interval is $1 - \left(x_i - \hat{\theta}\right)/2\epsilon$, and the probability mass at $L = 0$ is $\left(x_i - \hat{\theta}\right)/2\epsilon$.

Compared with marginal creditors who observe $x_i = \hat{\theta}$, creditors observing $x_i > \hat{\theta}$ hold more optimistic beliefs that a smaller proportion of wholesale creditors will withdraw (reflected by the positive probability mass placed on $L = 0$). As a marginal creditor who observes $x_i = \hat{\theta}$ is indifferent between 'withdraw' and 'wait', a creditor who observes $x_i > \hat{\theta}$ will strictly prefer 'wait'. Moreover, the higher the signal, the more optimistic the creditor is, so that $Prob\left(L = 0 | x_i, x_i > \hat{\theta}\right) = (x_i - \hat{\theta})/2\epsilon$ increases in x_i.

Case $x_i < \hat{\theta}$ can be analyzed by the same procedure. One can show that from the perspective of a creditor who observes $x_i < \hat{\theta}$, L again has a mixed distribution: it is uniformly distributed on $\left[(\hat{\theta} - x_i)/2\epsilon, 1\right]$ with density 1, and has a positive probability mass $(\hat{\theta} - x_i)/2\epsilon$ at $L = 1$. Thus, compared with a marginal creditor who observes $\hat{\theta}$, a creditor who observes $x_i < \hat{\theta}$ is more pessimistic and more likely to withdraw. Moreover, a creditor i becomes more pessmisitic when he receives a lower signal, i.e., $Prob(L = 1 | x_i, x_i > \hat{\theta}) = (\hat{\theta} - x_i)/2\epsilon$ decreasing in x_i.

Appendix A.3 Threshold equilibrium

We have shown that a creditor whose signal is higher (lower) than a critical $\hat{\theta}$ is more prone to wait (withdraw). Now we derive the value of this critical cash flow by studying the marginal creditors who observe $x_i = \hat{\theta}$. Such creditors perceive $L \sim U(0,1)$, and are indifferent between "withdraw" and "wait". Their indifference condition can be expressed as the following.

$$\int_0^1 DW(L)dL = 0$$

By the definition of $DW(L)$ and L^C (definitions in section 3.2), we can rewrite the indifference condition as follows.

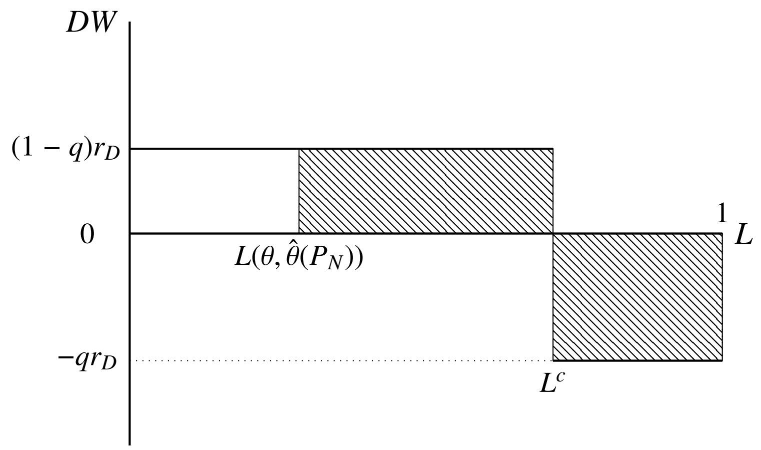
$$\int_0^{L^c} (1-q)r_D dL - \int_{L^c}^1 q r_D dL = (1-q)r_D L^c - q r_D(1-L^c) = 0$$

This implies a unique critical cash flow $\hat{\theta}$.

$$\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N}$$

For a given asset price $P_N \in [\underline{P}, D_2)$, runs happen to banks with $\theta < \hat{\theta}(P_N)$. Graphically, the indifference condition requires the two shaded areas to be equal in size, as shown in Figure 6.

Figure 6: Payoff differences and the decision to withdraw



Appendix B Proofs to Lemmas, propositions, and corollaries

Appendix B.1 Proposition 2. Solution to the baseline model

Proof. Step 1: Recall that the equilibrium condition (12) is a quadratic function of P_e. Solving it using the quadratic formula, we obtain the following two roots.[24]

$$P_e^- = \frac{\Psi - \sqrt{\Psi^2 - 8qD_1\underline{\theta}}}{4} \quad \text{and} \quad P_e^+ = \frac{\Psi + \sqrt{\Psi^2 - 8qD_1\underline{\theta}}}{4}$$

Step 2: We show that only P_e^+ can be part of an equilibrium. First, one can verify that both roots monotonically increase in q. In particular, one can show that $\partial P_e^-/\partial q$ increases in $\underline{\theta}$, and equals zero when $\underline{\theta} = 0$. Therefore, $\partial P_e^-/\partial q \geq 0$ for $\forall \underline{\theta} \geq 0$. Similarly, one can show that $\partial P_e^+/\partial q$ decreases in $\underline{\theta}$, and is greater than zero for $\forall \underline{\theta} < (2q-1)D_2$. Since $\underline{\theta} < (2q-1)D_2$ is guaranteed by assumption (4), we prove $\partial P_e^+/\partial q > 0$ as well.

Second, one can verify that P_e^+ is equal to $\underline{P}$ when $q = 0$, and is smaller than D_2 when $q = 1$. Whereas P_e^- is equal to 0 when $q = 0$, and is smaller than D_1 when $q = 1$. We have shown in section 3.1 that the equilibrium asset price must be between $\underline{P}$ and D_2. As $D_1 < \underline{P}$, P_e^- cannot be part of an equilibrium. Therefore, we retain only P_e^+, and obtain the following closed-form solution for the equilibrium asset price.

$$P_e = \frac{\Psi + \sqrt{\Psi^2 - 8qD_1\underline{\theta}}}{4}$$

The equilibrium critical cash flow can be solved from zero profit condition $P_e = (\theta_e + \underline{\theta})/2$.

$$\theta_e = \frac{\Psi + \sqrt{\Psi^2 - 8qD_1\underline{\theta}} - 2\underline{\theta}}{2}$$

Step 3: We prove that the equilibrium θ_e is indeed between θ^L and $\theta^U(P_e)$. First, one can verify that $\theta_e > \theta^L = D_2$. By the closed-form solution of θ_e, the inequality can be explicitly written as $\sqrt{\Psi^2 - 8qD_1\underline{\theta}} > 2D_2 + 2\underline{\theta} - \Psi$. To see this is true, one can square both sides and rearrange the terms to get $(D_2 + \underline{\theta})\Psi - (D_2 + \underline{\theta})^2 - 2qD_1\underline{\theta} > 0$. By the definition of Ψ, the inequality further simplify to $2qD_2 - D_2 - \underline{\theta} > 0$, which always holds by assumption (4).

[24] The parametric assumptions guarantee that $\Psi^2 - 8qD_1\underline{\theta}$ is always positive.

Finally, we show that $\theta_e < \theta^U(P_e)$. Note that when the asset market is in its equilibrium, it holds that $\theta_e = 2P_e - \underline{\theta}$. Therefore, to show $\theta_e < \theta^U(P_e)$ is equivalent to show

$$\frac{F}{1 - D_1/P_e} - (2P_e - \underline{\theta}) > 0,$$

or $2P_e^2 - (2D_1 + \underline{\theta} + F)P_e + D_1\underline{\theta} < 0$. Using the closed-form solution of P_e, one can rewrite the inequality as follows.

$$(1-q)D_1 \left[\frac{1-2q}{q} \frac{\Psi + \sqrt{\Psi^2 - 9qD_1\underline{\theta}}}{4} + \underline{\theta} \right] < 0 \tag{B.29}$$

One can verify that the part in the square bracket increases in $\underline{\theta}$, and equals zero when $\underline{\theta} = (2q-1)D_2$. By assumption (4), we know that $\underline{\theta} < (2q-1)D_2$. Therefore, the part in the square bracket must be negative, and this concludes the proof that $\theta_e < \theta^U(P_e)$.

Note that this proof does not rely on a specific value of $\underline{\theta}$. Therefore, the result applies to cases where $\underline{\theta}$ takes the value of $\underline{\theta}_B$, or $\underline{\theta}_G$, or their (weighted) average. □

Appendix B.2 Proposition 3. Bank capital and illiquidity

Proof. First, we show that once asset price is endogenous, increasing capital does not reduce the risk of bank run if $\underline{\theta} = 0$. To see so, note that using the closed-form solution of P_e, one can rewrite expression (18) as follows.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} \frac{(P_e - qD_1)\sqrt{\Psi^2 - 8qD_1\underline{\theta}} - (D_2 - D_1)P_e}{(P_e - qD_1)\sqrt{\Psi^2 - 8qD_1\underline{\theta}}}$$

Denote the numerator of the second fraction by $H(\underline{\theta}) \equiv (P_e - qD_1)\sqrt{\Psi^2 - 8qD_1\underline{\theta}} - (D_2 - D_1)P_e$. Note that when $\underline{\theta} = 0$, $H(\underline{\theta}) = qD_1\Psi$ and $P_e = \left[(D_2 - D_1) + 2qD_1\right]/2$. Substituting in this expression of P_e, we have $H(\underline{\theta}) = 0$. Therefore, for $\underline{\theta} = 0$, we obtain $\partial IL/\partial \Delta = 0$: increasing capital has no effect on the risk of bank runs.

On the other hand, when $\underline{\theta} > 0$, it can be proved that $H(\underline{\theta})$ is positive, so that capital reduces the risk of bank runs. Using the definition of Ψ and P_e, one can rewrite $H(\underline{\theta})$ as follows.

$$H(\underline{\theta}) = (\underline{\theta} - 2qD_1)(P_e - qD_1) + qD_1(D_2 - D_1)$$

We already know that $H(\underline{\theta}) = 0$ when $\underline{\theta} = 0$. Therefore, to prove that $H(\underline{\theta}) > 0$ for $\underline{\theta} > 0$, it is sufficient to show $\partial H/\partial\underline{\theta} > 0$. Note that $\partial P_e/\partial\underline{\theta} = (P_e - qD_1)/\sqrt{\Psi^2 - 8qD_1\underline{\theta}}$. We have

$$\begin{aligned}\frac{\partial H}{\partial \underline{\theta}} &= (P_e - qD_1) + (\underline{\theta} - 2qD_1)\frac{\partial P_e}{\partial \underline{\theta}} \\ &= (P_e - qD_1)\frac{(\underline{\theta} - 2qD_1) + \sqrt{\Psi^2 - 8qD_1\underline{\theta}}}{\sqrt{\Psi^2 - 8qD_1\underline{\theta}}}.\end{aligned}$$

Finally, by verifying that $(\underline{\theta} - 2qD_1) + \sqrt{\Psi^2 - 8qD_1\underline{\theta}} > 0$, we prove $H(\underline{\theta}) > 0$ for $\underline{\theta} > 0$. □

Appendix B.3 Lemma 2. The monotonicity of $\Pi_1(P_1)$

Proof. Recall that $\Pi_1(P_1)$, as defined in (20), represents asset buyers' expected payoff when they have observed a *single* bank run and purchase bank assets at price P_1. One can re-write $\Pi_1(P_1)$ as follows.

$$\Pi_1(P_1) = \frac{\hat{\theta}(P_1) + \underline{\theta}_G}{2} - \frac{\underline{\theta}_G - \underline{\theta}_B}{2}\omega_1^B(P_1) - P_1$$

Take the first order derivative with respect to P_1, we have

$$\frac{\partial \Pi_1}{\partial P_1} = \frac{1}{2}\frac{\partial\hat{\theta}(P_1)}{\partial P_1} - \frac{\underline{\theta}_G - \underline{\theta}_B}{2}\frac{\partial \omega_1^B(P_1)}{\partial P_1} - 1. \tag{B.30}$$

It is straightforward to see that $\partial\hat{\theta}(P_1)/\partial P_1 < 0$. Furthermore, one can verify that

$$\frac{\partial \omega_1^B(P_1)}{\partial P_1} = -\frac{\partial\hat{\theta}(P_1)}{\partial P_1}\frac{\underline{\theta}_G - \underline{\theta}_B}{\left[\left(\hat{\theta}(P_1) - \underline{\theta}_B\right) + \left(\hat{\theta}(P_1) - \underline{\theta}_G\right)\right]^2} > 0.$$

With all three terms of expression (B.30) being negative, we can, therefore, conclude that $\Pi_1(P_1)$ monotonically decreases in P_1.

In fact, the three terms of expression (B.30) are closely related to the intuition we discussed in section 4.1. Here, $\partial\hat{\theta}(P_1)/\partial P_1 < 0$ captures the fact that when asset buyers offer a higher price, the bank is more likely to sell its asset for insolvency, therefore, the average quality of the asset pool goes down. Whereas $-\partial\omega_1^B/\partial P_1 < 0$ captures the fact that when the bank run risk is mitigated by a higher asset price, State B is more likely given that a bank run is observed, which reduces the expected payoff of asset buyers as well. Finally, the last term captures the increase in the extra cost for acquiring the bank's assets. □

Appendix B.4 Proposition 4. The existence and uniqueness of $\{P_1^*, \theta_1^*\}$

Proof. The proof takes two steps. We first show the existence and uniqueness of a $P_1^* \in (\underline{P}, D_2)$, and then show that the corresponding θ_1^* is indeed between θ^L and $\theta^U(P_1^*)$.

Step 1: We prove by continuity that there exists a unique $P_1^* \in (\underline{P}, D_2)$ such that $\Pi_1(P_1^*) = 0$. Note that the equilibrium condition can be written explicitly as the following.

$$\Pi_1(P_1^*) = \omega_1^B\left(\hat{\theta}(P_1^*)\right)\pi^B\left(\hat{\theta}(P_1^*)\right) + \omega_1^G\left(\hat{\theta}(P_1^*)\right)\pi^G\left(\hat{\theta}(P_1^*)\right) = 0 \tag{B.31}$$

Here $\pi^s(P_1) \equiv \left(\underline{\theta}_s + \hat{\theta}(P_1)\right)/2 - P_1$ denotes a buyer's profit in State s when her offered price is P_1. One can show that, independently of State s, the buyer makes a profit if $P_1 = \underline{P}$. To see so, note that a sufficient condition for $\pi^s(\underline{P}) > 0$ is $\hat{\theta}(\underline{P}) > D_2$, which is implied by assumption (4).

$$\pi^s(\underline{P}) = \frac{\hat{\theta}(\underline{P}) + \underline{\theta}_s}{2} - \underline{P} = \frac{\hat{\theta}(\underline{P}) + \underline{\theta}_s}{2} - \frac{D_2 + \underline{\theta}_B}{2} > 0$$

On the other hand, one can show that $\hat{\theta}(D_2) < D_2$ for $\forall q < 1$, so that the buyer makes a loss if asset price $P_1 = D_2$.

$$\pi^s(D_2) = \frac{\hat{\theta}(D_2) + \underline{\theta}_s}{2} - D_2 < \frac{D_2 + \underline{\theta}_s}{2} - D_2 < 0$$

With the posterior beliefs $\omega_1^B\left(\hat{\theta}(P_1)\right)$ and $\omega_1^G\left(\hat{\theta}(P_1)\right)$ positive and smaller than one,[25] we know that $\Pi_1(P_1)$ is positive when $P_1 = \underline{P}$ and negative when $P_1 = D_2$. As $\Pi_1(P_1)$ monotonically decreases in P_1, we know by continuity that there exists a unique equilibrium $P_1^* \in (\underline{P}, D_2)$ such that $\Pi_1(P_1^*) = 0$.

Step 2: We now prove that the corresponding $\theta_1^* = \hat{\theta}(P_1^*)$ is indeed between θ^L and $\theta^U(P_1^*)$. In fact, for any number of bank runs, $N = 1, 2, 3, ...,$ denoting the equilibrium price associated with N bank runs by P_N^*, we have $\theta^L < \theta_N^*(P_N^*) < \theta^U(P_N^*)$.

We start with the second inequality $\theta_N^*(P_N^*) < \theta^U(P_N^*)$. First, note that the equilibrium price P_N^* should be characterized by equation (9).

$$P_N^* = \omega_N^B\left(\hat{\theta}(P_N^*)\right)\frac{\underline{\theta}_B + \hat{\theta}(P_N^*)}{2} + \omega_N^G\left(\hat{\theta}(P_N^*)\right)\frac{\underline{\theta}_G + \hat{\theta}(P_N^*)}{2} \tag{B.32}$$

[25]This result follows from $\hat{\theta}(P_1) > D_2 > \underline{\theta}_G > \underline{\theta}_B$, which we show in Step 2 of the proof.

Here, we have

$$\omega_N^B = \frac{\left(\hat{\theta}(P_N^*) - \underline{\theta}_B\right)^N}{\left(\hat{\theta}(P_N^*) - \underline{\theta}_B\right)^N + \left(\hat{\theta}(P_N^*) - \underline{\theta}_G\right)^N}$$

as a general form of buyers' posterior beliefs for $s = B$, when N bank runs are observed. Defining $\underline{\theta}_N(P_N^*) \equiv \omega_N^B\left(\hat{\theta}(P_N^*)\right) \cdot \underline{\theta}_B + \omega_N^G\left(\hat{\theta}(P_N^*)\right) \cdot \underline{\theta}_G$ and $\Psi(P_N^*) \equiv (D_2 - D_1) + 2qD_1 + \underline{\theta}_N(P_N^*)$, we can re-write equation (B.32) as follows.

$$2P_N^{*2} - \Psi(P_N^*)P_N^* + qD_1\underline{\theta}_N(P_N^*) = 0 \tag{B.33}$$

Further notice that, by the equilibrium condition (B.32), $\theta_N^*(P_N^*) < \theta^U(P_N^*)$ is equivalent to

$$\theta^U(P_N^*) = \frac{F}{1 - D_1/P_N^*} > \hat{\theta}(P_N^*) = 2P_N^* - \underline{\theta}_N(P_N^*),$$

or, alternatively,

$$2P_N^{*2} - \left[2D_1 + \underline{\theta}_N(P_N^*) + F\right]P_N^* + D_1\underline{\theta}_N(P_N^*) < 0. \tag{B.34}$$

Substituting (B.33) into (B.34) and using the defition of $\Psi(P_N^*)$, we obtain the following sufficient and necessary condition for $\theta_N^*(P_N^*) < \theta^U(P_N^*)$.[26]

$$(1-q)D_1\left[\frac{1-2q}{q}P_N^* + \underline{\theta}(P_N^*)\right] < 0$$

Let $C_N \equiv (1-2q)P_N^*/q + \underline{\theta}(P_N^*)$, we show that C_N is negative for $N = 1, 2, 3, ...$ To see so, first notice that $C_N < C_{N+1}$ for $N = 1, 2, 3, ...$ In particular, we have

$$\underline{\theta}_N(P_N^*) < \underline{\theta}_{N+1}(P_N^*) < \underline{\theta}_{N+1}(P_{N+1}^*).$$

The first inequality follws from the fact that, for a given price P, $\omega_N^B(P)$ decreases in N, and the second inquality follows from the fact that, for a given N, $\omega_N^B(P)$ increases in P. Second, one can show that $C_N < 0$ holds for $N = 0$ and $N \to +\infty$. In fact, these two cases reduce to a scenario where the posterior beliefs on State s do not depend on the asset price. We have $\omega_0^B = 1/2$ in the former case, and $\omega_{+\infty}^B \to 1$ in the latter case. As a result, the two cases degenerate to the baseline model we analyzed in Appendix B.1, and the analysis on inequality

[26] This condition is a counterpart of inequality (B.29).

(B.29) directly applies. Therefore, we prove $C_0 < 0$ and $C_{+\infty} < 0$. Together with the fact that C_N increases in N, we can conclude that $C_N < 0$ for $N = 1, 2, 3, ...$

Finally, it takes two steps to prove $\theta^L < \theta^*_N$. First, one can show that even if the asset buyers hold an extremely optimistic belief $\omega^G_N = 1$, there still exists a corresponding critical cash flow $\theta^*_G > \theta^L$ (Lemma 6). Second, one can show that $\theta^*_1 > \theta^*_G$, because of buyers' more pessimistic beliefs about State s (Proposition 8). In fact, one can show that $\theta^L < \theta^*_N$ is generally true for any θ^*_N, $N = 1, 2, 3, ...$ (Lemma 4). □

Appendix B.5 Lemma 3. The monotonicity of $\Pi_2(P_2)$

Proof. The proof resembles that of Lemma 2. One can re-write $\Pi_2(P_2)$ as

$$\Pi_2(P_2) = \frac{\hat{\theta}(P_2) + \underline{\theta}_G}{2} - \frac{\underline{\theta}_G - \underline{\theta}_B}{2}\omega^B_2(P_2) - P_2,$$

and obtain its first order derivative with respect to P_2,

$$\frac{\partial \Pi_2}{\partial P_2} = \frac{1}{2}\frac{\partial\hat{\theta}(P_2)}{\partial P_2} - \frac{\underline{\theta}_G - \underline{\theta}_B}{2}\frac{\partial\omega^B_2(P_2)}{\partial P_2} - 1. \tag{B.35}$$

Again, one can prove that the belief about State s deteriorates with a higher asset price.

$$\frac{\partial\omega^B_2(P_2)}{\partial P_2} = -2\frac{\partial\hat{\theta}(P_2)}{\partial P_2}\frac{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)\left(\hat{\theta}(P_2) - \underline{\theta}_G\right)\left(\underline{\theta}_G - \underline{\theta}_B\right)}{\left[\left(\hat{\theta}(P_1) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_1) - \underline{\theta}_G\right)^2\right]^2} > 0$$

With all three components of expression (B.35) being negative, we can conclude thtat $\Pi_2(P_2)$ monotonically decreases in P_2. □

Appendix B.6 Proposition 5. The existence and uniqueness of $\{P^*_2, \theta^*_2\}$

Proof. The proof follows the same argument as in Appendix B.4. Similar to that proof, the equilibrium condition can be expressed as follows.

$$\Pi_2(P^*_2) = \omega^B_2\left(\hat{\theta}(P^*_2)\right)\pi^B\left(\hat{\theta}(P^*_2)\right) + \omega^G_2\left(\hat{\theta}(P^*_2)\right)\pi^G\left(\hat{\theta}(P^*_2)\right) = 0 \tag{B.36}$$

It still holds that, independent of State s, buyers make a profit when the asset price is $\underline{P}$, and a loss when the asset price is D_2, i.e., $\pi^s\left(\hat{\theta}(\underline{P})\right) > 0$ and $\pi^s\left(\hat{\theta}(D_2)\right) < 0$. Furthermore, it is straightforward to verify that the posterior beliefs $\omega_2^G\left(\hat{\theta}(P_2)\right)$ and $\omega_2^G\left(\hat{\theta}(P_2)\right)$ are both positive and smaller than 1.

As a result, we prove that $\Pi_2(P_2)$ is positive when $P_2 = \underline{P}$, and negative when $P_2 = D_2$. Given that $\Pi_2(P_2)$ monotonically decreases in P_2, we know by continuity that there exists a unique equilibrium P_2^* so that $\Pi_2(P_2^*) = 0$.

Finally, by the second half of Appendix B.4, we know that $\theta_2^* \in (\theta^L, \theta^U(P_2^*))$. □

Appendix B.7 Lemma 4. Financial contagion

Proof. Let $P(\hat{\theta})$ be the inverse function of $\hat{\theta}(P)$. For any given critical cash flow $\hat{\theta}$, the proof hinges on the monotonicity of two ratios.

$$\frac{\omega_2^B(\hat{\theta})}{\omega_2^G(\hat{\theta})} = \frac{(\hat{\theta} - \underline{\theta}_B)^2}{(\hat{\theta} - \underline{\theta}_G)^2} \quad \text{and} \quad \frac{\pi^G(\hat{\theta})}{\pi^B(\hat{\theta})} = \frac{(\hat{\theta} + \underline{\theta}_G)/2 - P(\hat{\theta})}{(\hat{\theta} + \underline{\theta}_B)/2 - P(\hat{\theta})}$$

The former is a conditional likelihood ratio, and the latter is a payoff ratio. Since $\partial P(\hat{\theta})/\partial\hat{\theta} < 0$, it can be shown that both ratios strictly monotonically decrease in $\hat{\theta}$ for $\hat{\theta} > D_2$.

$$\frac{d}{d\hat{\theta}}\left(\frac{\omega_2^B(\hat{\theta})}{\omega_2^G(\hat{\theta})}\right) = -\frac{2(\hat{\theta} - \underline{\theta}_B)(\underline{\theta}_G - \underline{\theta}_B)}{\left(\hat{\theta} - \underline{\theta}_G\right)^3} < 0$$
$$\frac{d}{d\hat{\theta}}\left(\frac{\pi^G(\hat{\theta})}{\pi^B(\hat{\theta})}\right) = -\frac{\left[1/2 - \partial P(\hat{\theta})/\partial\hat{\theta}\right](\underline{\theta}_G - \underline{\theta}_B)}{2\left[\frac{\hat{\theta}+\underline{\theta}_B}{2} - P(\hat{\theta})\right]^2} < 0$$

Furthermore, notice that $\omega_1^B(\hat{\theta})/\omega_1^G(\hat{\theta}) > 1$, $\left[\omega_1^B(\hat{\theta})/\omega_1^G(\hat{\theta})\right]^2 = \omega_2^B(\hat{\theta})/\omega_2^G(\hat{\theta})$, and therefore

$$\frac{\omega_1^B(\hat{\theta})}{\omega_1^G(\hat{\theta})} < \left[\frac{\omega_1^B(\hat{\theta})}{\omega_1^G(\hat{\theta})}\right]^2 = \frac{\omega_2^B(\hat{\theta})}{\omega_2^G(\hat{\theta})}. \tag{B.37}$$

We prove by contradiction. Suppose $\theta_1^* \geq \theta_2^*$. By the monotonicity of $\pi^G(\hat{\theta})/\pi^B(\hat{\theta})$, we have

$$\frac{\pi^G(\theta_1^*)}{\pi^B(\theta_1^*)} \leq \frac{\pi^G(\theta_2^*)}{\pi^B(\theta_2^*)}. \tag{B.38}$$

By equilibrium conditions (B.31) and (B.36), we have

$$\frac{\pi^G(\theta_1^*)}{\pi^B(\theta_1^*)} = -\frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} \quad \text{and} \quad \frac{\pi^G(\theta_2^*)}{\pi^B(\theta_2^*)} = -\frac{\omega_2^B(\theta_2^*)}{\omega_2^G(\theta_2^*)}.$$

Together with (B.38), the two equations above imply that

$$\frac{\omega_2^B(\theta_2^*)}{\omega_2^G(\theta_2^*)} \leq \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)}.$$

By inequality (B.37), we know

$$\frac{\omega_2^B(\theta_2^*)}{\omega_2^G(\theta_2^*)} \leq \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} < \frac{\omega_2^B(\theta_1^*)}{\omega_2^G(\theta_1^*)}.$$

But this contradicts the strict monotonicity of $\omega_2^B(\hat{\theta})/\omega_2^G(\hat{\theta})$. Therefore, we prove $\theta_2^* > \theta_1^*$. In fact, one can modify this proof to show that $\theta_{N+1}^* > \theta_N^*$, $N = 1, 2, 3, ...$ is generally true. □

Appendix B.8 Lemma 5. Regulator's break-even price P_A^*

Proof. The proof resembles that in Appendix B.1. Inserting the expression of $\hat{\theta}(P_A)$ into the regulator's break-even condition (24), one can obtain the following quadratic equation of P_A^*.

$$2P_A^{*2} - \overline{\Psi} P_A^* + qD_1 E(\underline{\theta}) = 0$$

The equation resembles (12) and has two roots. Following the proof in Appendix B.1, one can verify that both roots increase in q, and that only the root with the positive sign

$$P_A^* = \frac{\overline{\Psi} + \sqrt{\overline{\Psi}^2 - 8qD_1 E(\underline{\theta})}}{4}$$

falls in the interval $[\underline{P}, D_2)$. Therefore, we have a unique break-even price P_A^* for the regulator. It also takes the same procedure as in Appendix B.1 to show $\theta_A^* \in (\theta^L, \theta^U(P_A^*))$. □

Appendix B.9 Proposition 7. The impact of asset purchase programs

Proof. We denote by Π_A the regulator's expected payoff when the regulator commits to buy bank assets at a price P_A. With $\hat{\theta}_A \equiv \hat{\theta}(P_A)$, Π_A can be written explicitly as follows.

$$\Pi_A(\hat{\theta}_A) = \frac{1}{2}\frac{\hat{\theta}_A + \underline{\theta}_B}{2} + \frac{1}{2}\frac{\hat{\theta}_A + \underline{\theta}_G}{2} - \frac{qD_1\hat{\theta}_A}{\hat{\theta}_A - (D_2 - D_1)}$$

Denote $\hat{\theta}_1 \equiv \hat{\theta}(P_1)$, Π_1 can be written compactly as follows.

$$\Pi_1 = \frac{\hat{\theta}_1 - \underline{\theta}_B}{(\hat{\theta}_1 - \underline{\theta}_B) + (\hat{\theta}_1 - \underline{\theta}_G)} \frac{\hat{\theta}_1 + \underline{\theta}_B}{2} + \frac{\hat{\theta}_1 - \underline{\theta}_G}{(\hat{\theta}_1 - \underline{\theta}_B) + (\hat{\theta}_1 - \underline{\theta}_G)} \frac{\hat{\theta}_1 + \underline{\theta}_G}{2} - \frac{qD_1\hat{\theta}_1}{\hat{\theta}_1 - (D_2 - D_1)} \tag{B.39}$$

Evaluating $\Pi_1(\hat{\theta}_1)$ at θ_A^*, we have

$$\begin{aligned}
\Pi_1(\theta_A^*) &= \frac{1}{2}\frac{\underline{\theta}_B + \theta_A^*}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \theta_A^*}{2} - \frac{qD_1\theta_A^*}{\theta_A^* - (D_2 - D_1)} - \frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4\left[(\theta_A^* - \underline{\theta}_G) + (\theta_A^* - \underline{\theta}_B)\right]} \\
&= \Pi_A(\theta_A^*) - \frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4\left[(\theta_A^* - \underline{\theta}_G) + (\theta_A^* - \underline{\theta}_B)\right]}.
\end{aligned}$$

The last term reflects deteriorating beliefs about State s. By its definition, θ_A^* makes $\Pi_A(\theta_A^*) = 0$. Therefore, we have

$$\Pi_1(\theta_A^*) = -\frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4\left[(\theta_A^* - \underline{\theta}_G) + (\theta_A^* - \underline{\theta}_B)\right]} < 0.$$

Since $\Pi_1(\hat{\theta}_1)$ increases in $\hat{\theta}_1$ and $\Pi_1(\theta_1^*) = 0$, we prove $\theta_A^* < \theta_1^*$. The result $P_A^* > P_1^*$ immediately follows. □

Appendix B.10 Lemma 6. Regulatory disclosure and bank runs

Proof. The proof follows that of Proposition 2. When the asset buyers know for sure that the realized state is s, $s \in \{B, G\}$, the equilibrium θ_s^* and P_s^* are determined by a system of two equations: the threshold equilibrium in the bank run game,

$$\theta_s^* = \frac{D_2 - D_1}{1 - qD_1/P_s^*},$$

and the zero-profit condition in the secondary asset market,

$$P_s^* = \frac{\theta_s^* + \underline{\theta}_s}{2}.$$

For each State $s \in \{B, G\}$, one can obtain a quadratic equation of P_s^*, and solving it leads to two roots. As only the one with the positive sign falls between $\underline{P}$ and D_2, we can obtain closed-form solutions for P_s^* and corresponding θ_s^*.

$$P_s^* = \frac{\Psi_s + \sqrt{\Psi_G^2 - 8qD_1\underline{\theta}_s}}{4}$$
$$\theta_s^* = \frac{\Psi_s + \sqrt{\Psi_G^2 - 8qD_1\underline{\theta}_s} - 2\underline{\theta}_s}{2}$$

Here $\Psi_s \equiv (D_2 - D_1) + 2qD_1 + \underline{\theta}_s$. Following the same procedure in Appendix B.1, one can prove that $\theta_s^* \in (\theta^L, \theta^U(P_s^*))$. □

Appendix B.11 Proposition 8. Regulatory disclosure and illiquidity

Proof. The proof resembles that of Proposition 7. First, we prove $\theta_G^* < \theta_1^*$. Recall that $\Pi_1(\hat{\theta}_1)$—as defined in (B.39)—monotonically increases in $\hat{\theta}_1$ and equals zero when $\hat{\theta}_1 = \theta_1^*$. Therefore, $\theta_G^* < \theta_1^*$ will be true if and only if $\Pi_1(\theta_G^*) < 0$. Evaluating $\Pi_1(\hat{\theta}_1 s)$ at θ_G^*, we have

$$\begin{aligned}\Pi_1(\theta_G^*) &= \frac{\theta_G^* + \underline{\theta}_G}{2} - \frac{qD_1\theta_G^*}{\theta_G^* - (D_2 - D_1)} + \frac{\theta_G^* - \underline{\theta}_B}{(\theta_G^* - \underline{\theta}_B) + (\theta_G^* - \underline{\theta}_G)}\left[\frac{\theta_G^* + \underline{\theta}_B}{2} - \frac{\theta_G^* + \underline{\theta}_G}{2}\right] \\ &= \frac{\theta_G^* + \underline{\theta}_G}{2} - \frac{qD_1\theta_G^*}{\theta_G^* - (D_2 - D_1)} - \frac{\theta_G^* - \underline{\theta}_B}{(\theta_G^* - \underline{\theta}_B) + (\theta_G^* - \underline{\theta}_G)}\frac{\underline{\theta}_G - \underline{\theta}_B}{2}.\end{aligned}$$

Note that by the definition of θ_G^*, it holds that

$$\frac{\theta_G^* + \underline{\theta}_G}{2} - \frac{qD_1\theta_G^*}{\theta_G^* - (D_2 - D_1)} = \frac{\theta_G^* + \underline{\theta}_G}{2} - P_G^* = 0.$$

Â· Therefore, by evaluating $\Pi_1(\theta)$ at θ_G^*, we have

$$\Pi_1(\theta_G^*) = -\frac{\theta_G^* - \underline{\theta}_B}{(\theta_G^* - \underline{\theta}_B) + (\theta_G^* - \underline{\theta}_G)}\frac{\underline{\theta}_G - \underline{\theta}_B}{2} < 0,$$

which in turn implies $\theta_G^* < \theta_1^*$.

Now, following a similar procedure and denoting $\hat{\theta}_2 \equiv \hat{\theta}(P_2)$, we prove $\theta_B^* > \theta_2^*$. Recall that $\Pi_2(\hat{\theta}_2)$ monotonically increases in $\hat{\theta}_2$ and equals zero when $\hat{\theta}_2 = \theta_2^*$. Therefore, $\theta_B^* > \theta_2^*$ will hold if and only if $\Pi_2(\theta_B^*) > 0$. Evaluating $\Pi_2(\hat{\theta}_2)$ at θ_B^*, we have

$$\Pi_2(\theta_B^*) = \frac{\theta_B^* + \underline{\theta}_B}{2} - \frac{qD_1\theta_B^*}{\theta_B^* - (D_2 - D_1)} + \frac{(\theta_B^* - \underline{\theta}_G)^2}{(\theta_B^* - \underline{\theta}_B)^2 + (\theta_B^* - \underline{\theta}_G)^2}\frac{\underline{\theta}_G - \underline{\theta}_B}{2}.$$

Note that by the definition of θ_B^*, it holds that

$$\frac{\theta_B^* + \underline{\theta}_G}{2} - \frac{qD_1\theta_B^*}{\theta_B^* - (D_2 - D_1)} = \frac{\theta_B^* + \underline{\theta}_B}{2} - P_B^* = 0.$$

Therefore, when evaluating $\Pi_2(\hat{\theta}_2)$ at θ_B^*, we have

$$\Pi_2(\theta_B^*) = \frac{(\theta_B^* - \underline{\theta}_G)^2}{(\theta_B^* - \underline{\theta}_B)^2 + (\theta_B^* - \underline{\theta}_G)^2} \frac{\underline{\theta}_G - \underline{\theta}_B}{2} > 0,$$

which in turn implies $\theta_B^* > \theta_2^*$. □

Appendix B.12 Corollary 1. Comparing policy interventions

Proof. It is straightforward to verify that $SC_{AP} < SC_{RD}$ is true if and only if

$$\frac{\theta_A^* - \theta_B^*}{\overline{\theta} - \underline{\theta}_B} < \frac{\theta_G^* - \theta_A^*}{\overline{\theta} - \underline{\theta}_G}. \tag{B.40}$$

To derive the critical condition in corollary 1 and verify the expression is positive, one needs to know $\theta_A^* < \left(\underline{\theta}_B + \underline{\theta}_G\right)/2$. To see so, recall that $\{\theta_A^*, P_A^*\}$ satisfies the threshold equilibrium (8) and the regulator's break even condition (24). Therefore, we have

$$2P_A^* = \frac{2qD_1\theta_A^*}{\theta_A^* - (D_2 - D_1)} = \theta_A^* + \frac{\underline{\theta}_B + \underline{\theta}_G}{2}. \tag{B.41}$$

Consider an auxiliary function

$$G(\theta) = \frac{2qD_1\theta}{\theta - (D_2 - D_1)} - \theta.$$

Then (B.41) can be written as

$$G(\theta_A^*) = \frac{\underline{\theta}_B + \underline{\theta}_G}{2}. \tag{B.42}$$

Similarly, for market equilibrium where when State s is known due to disclosure, we can obtain $G(\theta_B^*) = \underline{\theta}_B$, $G(\theta_G^*) = \underline{\theta}_G$, and

$$\frac{1}{2}G(\theta_B^*) + \frac{1}{2}G(\theta_G^*) = \frac{\underline{\theta}_B + \underline{\theta}_G}{2}. \tag{B.43}$$

Combining equation (B.42) and (B.43), we have the following equality.

$$G(\theta_A^*) = \frac{1}{2}G(\theta_B^*) + \frac{1}{2}G(\theta_G^*)$$

One can verify that for $\theta > D_2$, function $G(\theta)$ is decreasing and convex.

$$\frac{\partial G(\theta)}{\partial \theta} = -\frac{2qD_1(D_2 - D_1)}{[\theta - (D_2 - D_1)]^2} - 1 < 0 \quad \text{and} \quad \frac{\partial^2 G(\theta)}{\partial \theta^2} = \frac{4qD_1(D_2 - D_1)}{[\theta - (D_2 - D_1)]^3} > 0$$

By the convexity of $G(\theta)$, we have

$$G(\theta_A^*) = \frac{1}{2}G(\theta_B^*) + \frac{1}{2}G(\theta_G^*) > G\left(\frac{\theta_B^* + \theta_G^*}{2}\right)$$

Also, as function $G(\theta)$ decreases in θ, we obtain $\theta_A^* < \left(\underline{\theta}_B + \underline{\theta}_G\right)/2$.

With $\theta_A^* < \left(\underline{\theta}_B + \underline{\theta}_G\right)/2$, we can conclude that the social cost associated with bank failure is lower under asset purchase programs than under regulatory disclosures if and only if

$$\overline{\theta} > \frac{\underline{\theta}_G\theta_B^* + \underline{\theta}_B\theta_G^* - \left(\underline{\theta}_B + \underline{\theta}_G\right)\theta_A^*}{\theta_B^* + \theta_G^* - 2\theta_A^*} > 0.$$

□